Solving the Navier-Stokes Equations

Advanced CFD 02

NS Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\tau = \mu\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right) - \frac{2}{3}\mu[I]\nabla \cdot \mathbf{v}$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot \tau - \nabla p + \mathbf{B}$$

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho \mathbf{v} u) = \nabla \cdot (\mu \nabla u) - \frac{\partial p}{\partial x} + B^x$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho \mathbf{v} v) = \nabla \cdot (\mu \nabla v) - \frac{\partial p}{\partial y} + B^y$$

Incompressible flow: density is constant

variables: u,v, P

Highly non-linear

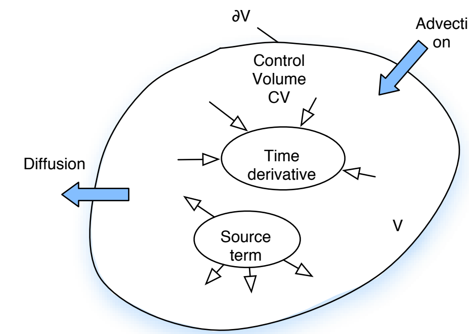
Momentum used for u and v

Pressure equation to be derived

$$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\textit{transient term}} + \underbrace{\nabla.(\rho\mathbf{v}\phi)}_{\textit{convection term}} = \underbrace{\nabla.(\Gamma\nabla\phi)}_{\textit{diffusion term}} + \underbrace{Q}_{\textit{Sources and other terms}}$$

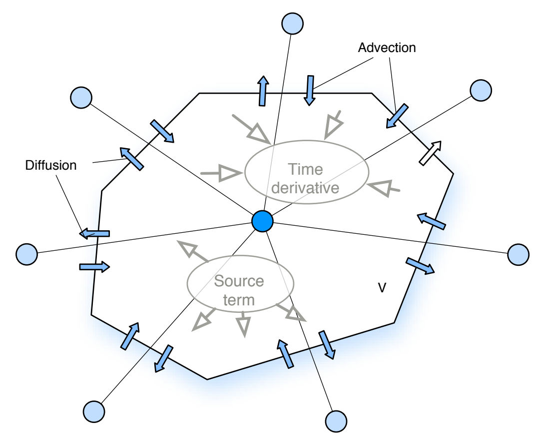
Discretization

$$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\text{transient term}} + \underbrace{\nabla.(\rho\mathbf{v}\phi)}_{\text{convection term}} = \underbrace{\nabla.(\Gamma\nabla\phi)}_{\text{diffusion term}} + \underbrace{Q}_{\text{Sources and other terms}}$$

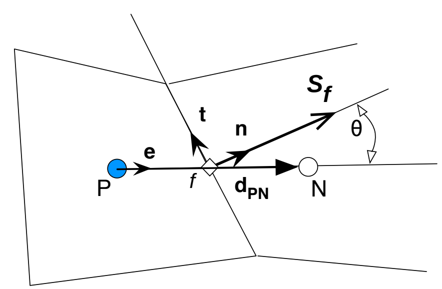


$$\int_V \frac{\partial(\rho\phi)}{\partial t} dV + \oint_{\partial V} (\rho\mathbf{v}\phi)\cdot d\mathbf{S} = \oint_{\partial V} (\Gamma\nabla\phi)\cdot d\mathbf{S} + \int_V Q dV$$

$$\frac{\partial(\rho\phi)}{\partial t} V_P + \sum_{f=nb(P)} (\rho\mathbf{v}\phi - \Gamma\nabla\phi)_f \cdot \mathbf{S}_f = Q_P V_P$$

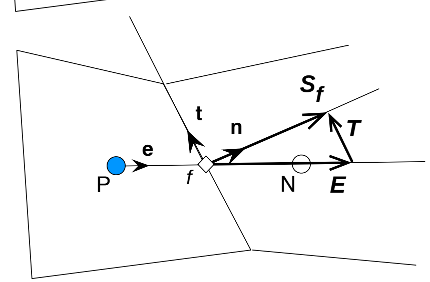


$$(\rho\mathbf{v}\phi)_f \cdot \mathbf{S}_f = \dot{m}_f \phi_f$$

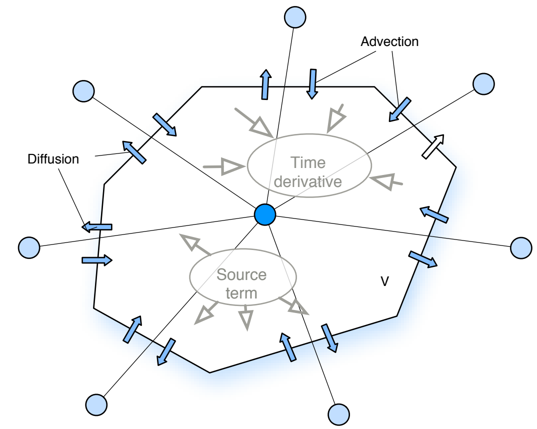


$$(-\Gamma\nabla\phi)_f \cdot \mathbf{S}_f = (-\Gamma\nabla\phi)_f \cdot \mathbf{E}_f + (-\Gamma\nabla\phi)_f \cdot \mathbf{T}_f$$

$$(-\Gamma\nabla\phi)_f \cdot \mathbf{S}_f = -\Gamma_f \left(\frac{\phi_F - \phi_P}{d_{PF}} E_f + (\nabla\phi)_f \cdot \mathbf{T}_f \right)$$



Algebraic Relation



$$\left(a_P^t + a_P^{CD}\right)\phi_P + \sum_{F=NB(P)} a_F^{CD}\phi_F = Q_P V_P + a_P^t \phi_P^\circ$$

$$\left(\frac{a_P^t + a_P^{CD}}{urf}\right)\phi_P + \sum_{\sim N=NB(P)} a_F^{CD}\phi_F = Q_P V_P + a_P^t \phi_P^\circ + \frac{(1-urf)}{urf}\left(a_P^t + a_P^{CD}\right)\phi_P^*$$

$$a_P\phi_P + \sum_{\sim N=NB(P)} \left(a_F\phi_F\right) = b_P$$

$$a_F = -\Gamma_f \frac{E_f}{d_{PF}} - \left\| -\dot{m}_f, 0 \right\| \qquad a_P = a_P^t - \sum_{F=NB(P)} a_F + \sum_{f=nb(P)} \dot{m}_f$$

$$b_P = Q_P V_P + a_P^t \phi_P^\circ + \frac{(1-urf)}{urf}\left(a_P^t + a_P^{CD}\right)\phi_P^* + \sum_{f=nb(P)} (\Gamma \nabla \phi)_f \cdot \mathbf{T}_f$$

$$\phi_P + \sum_{\sim F=NB(P)} \left(\frac{a_F}{a_P}\phi_F\right) = \frac{b_P}{a_P} \quad \Rightarrow \quad \phi_P + H_P[\phi] = B_P$$

for the Momentum equations

$$\mathbf{v_P} + \mathbf{H_P}[\mathbf{v}] = -\mathbf{D_P}(\nabla p)_P + \mathbf{B_P}$$

$$\mathbf{D_P} = \begin{bmatrix} d_P^u & 0 \\ 0 & d_P^v \end{bmatrix} = \begin{bmatrix} \dfrac{V_P}{a_P^u} & 0 \\ 0 & \dfrac{V_P}{a_P^v} \end{bmatrix}$$

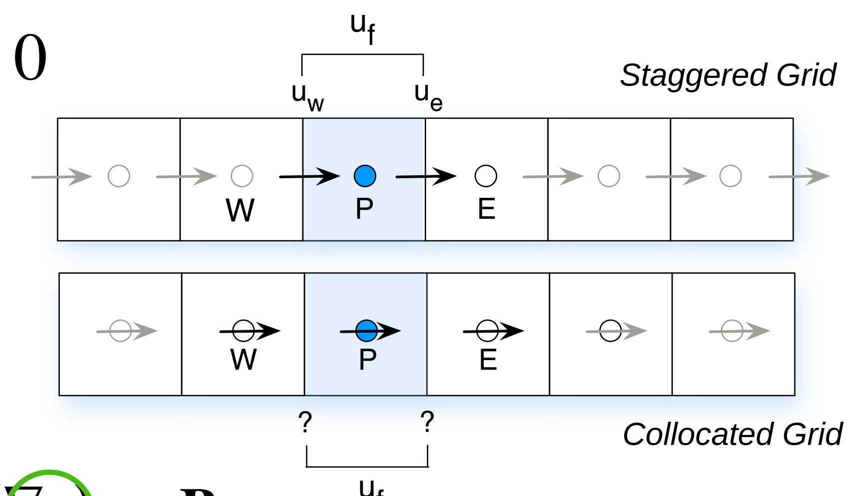
A Pressure Equation

$$\sum_{f=nb(P)} \dot{m}_f = 0 \ ?$$

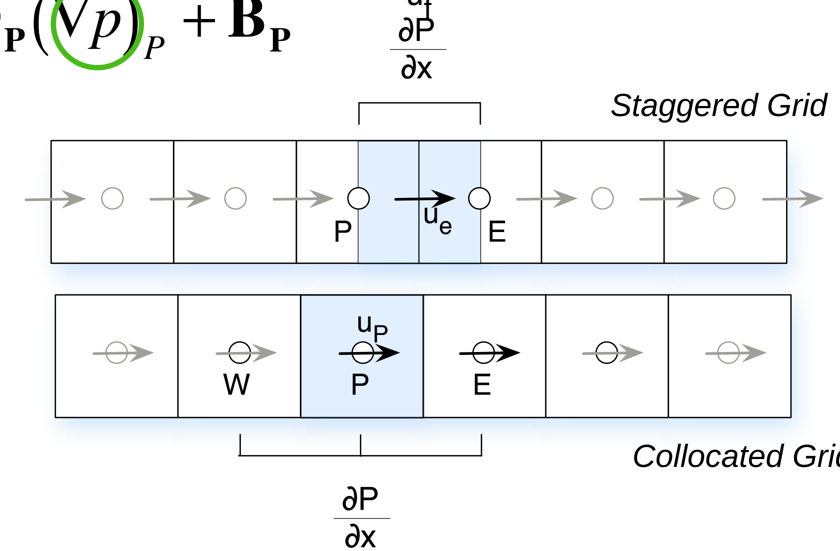
The Staggered grid

$$\sum_{f=nb(P)} \dot{m}_f = 0$$

$$\sum_{f=nb(P)} \dot{m}_f = (\rho \mathbf{v}\cdot\mathbf{S})_e + (\rho \mathbf{v}\cdot\mathbf{S})_w = (\rho_e u_e - \rho_w u_w)\Delta y$$



$$\mathbf{v_P} + \mathbf{H_P}[\mathbf{v}] = -\mathbf{D_P}(\nabla p)_P + \mathbf{B_P}$$



$$\left(\frac{\partial p}{\partial x}\right)_e = \frac{p_E - p_P}{(\delta x)_e} \quad and \quad \left(\frac{\partial P}{\partial x}\right)_w = \frac{p_P - p_W}{(\delta x)_w}$$

$$\left(\frac{\partial p}{\partial x}\right)_P = \frac{\tfrac{1}{2}(p_E + p_P) - \tfrac{1}{2}(p_P + p_W)}{(\Delta x)_P}$$

The Continuity Constraint

$$\mathbf{v}_f^* + \mathbf{H}_f[\mathbf{v}^*] = -\mathbf{D}_f^*\left(\nabla p^{(n)}\right)_f + \mathbf{B}_f$$

BUT

$$\sum_f \dot{m}_f^* \neq 0$$

$$\mathbf{v} = \mathbf{v}^* + \mathbf{v}' \qquad p = p^{(n)} + p'$$

$$\mathbf{v}_f + \mathbf{H}_f[\mathbf{v}] = -\mathbf{D}_f(\nabla p)_f + \mathbf{B}_f$$

AND

$$\sum_f \dot{m}_f = 0$$

$$\mathbf{v}'_f + \mathbf{H}_f[\mathbf{v}'] = -\mathbf{D}_f \nabla (p')_f$$

$$\sum_{f=nb(P)} \dot{m}_f = \sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}'_f\right) = \sum_{f=nb(P)} (\rho \mathbf{v}\cdot\mathbf{S})_f = \sum_{f=nb(P)} \left[\rho\left(\mathbf{v}^* + \mathbf{v}'\right)\cdot\mathbf{S}\right]_f = 0$$

$$\sum_{\sim f=nb(P)} \left(-\rho_f (\mathbf{D}\nabla p')_f \cdot \mathbf{S}_f\right) + \sum_{f=nb(P)} \dot{m}_f^* - \underline{\sum_{f=nb(P)} \left(\rho_f \mathbf{H}_f[\mathbf{v}'] \cdot \mathbf{S}_f\right) = 0}$$

The SIMPLE Algorithm

$$\underbrace{\sum_{f=nb(P)}\left[-\rho_f^{(n)}\mathbf{D}_f\left(\nabla p'\right)_f\cdot\mathbf{S}_f\right]}_{diffusion-like\ term} = -\underbrace{\sum_{f=nb(P)}\dot{m}_f^*}_{source-like\ term}$$

$$\sum_{f=nb(P)}\left[-\rho_f^{(n)}\mathbf{D}_f\left(\nabla p'\right)_f\cdot\mathbf{S}_f\right] = \sum_{f=nb(P)}\left[-\rho_f^{(n)}\mathbf{D}_f\left(\nabla p'\right)_f\cdot\left(\mathbf{E}_f+\mathbf{T}_f\right)\right]$$

$$\sum_{f=nb(P)}\left[-\rho_f^{(n)}\mathbf{D}_f\left(\nabla p'\right)_f\cdot\mathbf{E}_f\right] = \sum_{f=nb(P)}\left[-\rho_f^{(n)}\mathcal{D}_\mathsf{f}\left(p'_F-p'_P\right)\right]$$

$$\mathcal{D}_\mathsf{f} = \left(\frac{\overline{d_f^u}E_{x,f}+\overline{d_f^v}E_{y,f}}{\left\|\mathbf{d}_{PF}\right\|}\right)$$

The Pressure-Correction Equation

$$\dot{m}_f^* = \rho_f \mathbf{v}_f^* \cdot \mathbf{S}_f$$

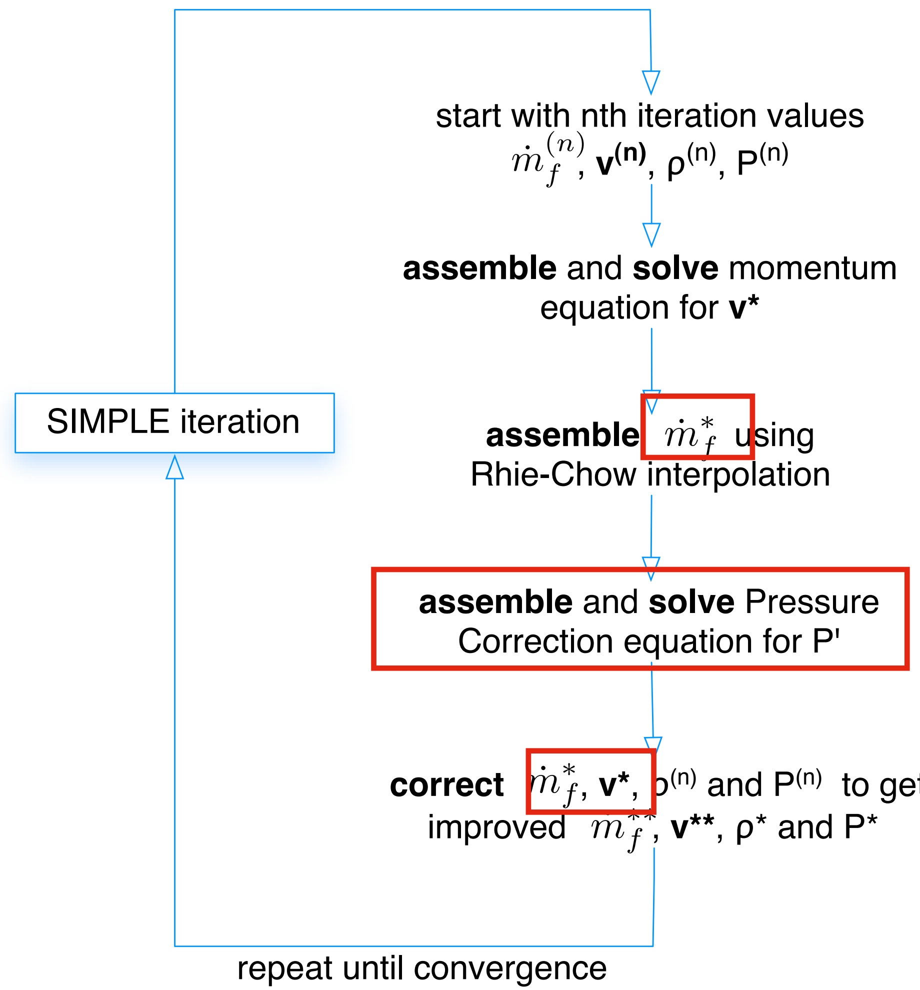
$$a_P p_P' + \sum_{F=nb(P)} a_F p_F' = b_P$$

$$b_P = -\sum_{f=nb(P)} \dot{m}_f^* + \sum_{f=nb(P)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p_f'\right) \cdot \mathbf{T}_f$$

$$a_F = -\rho_f^{(n)} \mathcal{D}_f$$

$$a_P = \sum_{f=nb(P)} \rho_f^{(n)} \mathcal{D}_f$$

$$\mathbf{v}_f' = -\mathbf{D}_f \nabla p_f'$$



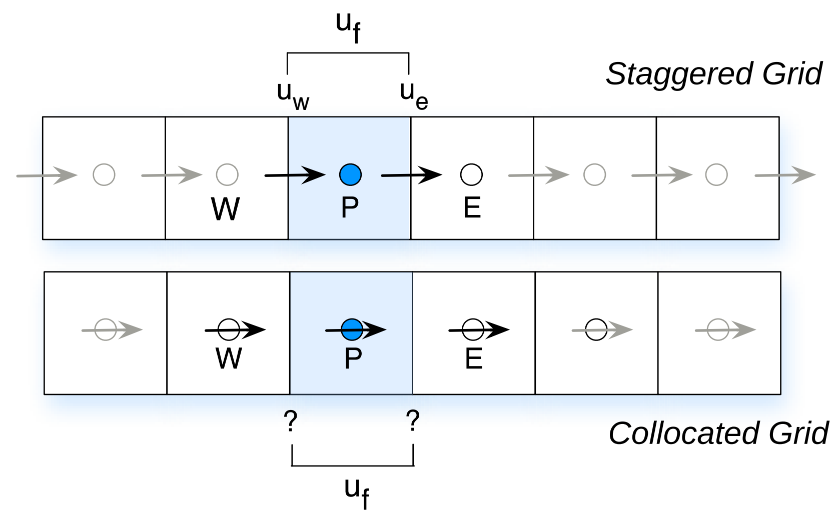
Collocated Grid

$$\sum_{f=nb(P)}\left(\dot{m}_f^* + \dot{m}_f'\right) = 0$$

$$u_P + H_P[u] = -D_P\left(\frac{\partial p}{\partial x}\right)_P$$

$$u_F + H_F[u] = -D_F\left(\frac{\partial p}{\partial x}\right)_F$$

$$u_f + H_f[u] = -D_f\left(\frac{\partial p}{\partial x}\right)_f$$



$$u_f + \overline{H_f}[u] = -\overline{D_f}\left(\frac{\partial p}{\partial x}\right)_f$$

$$D_f = \frac{1}{2}(D_P + D_F) = \overline{D_f}$$

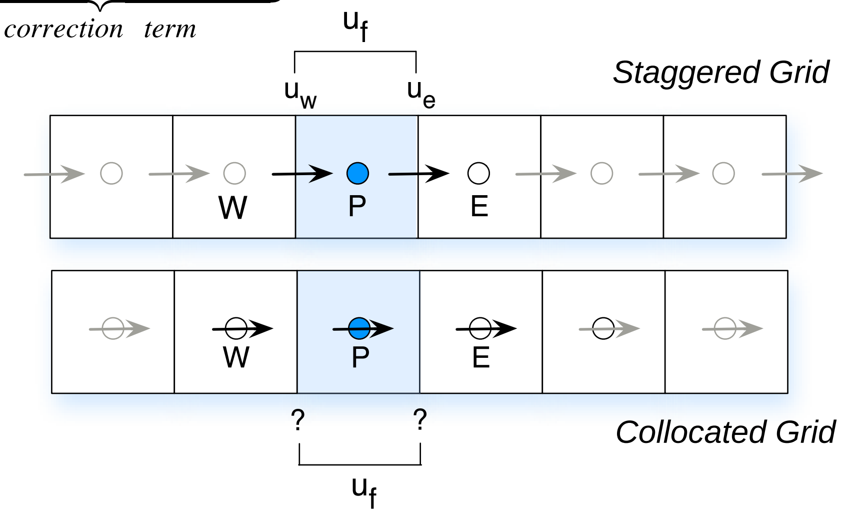
$$H_f[u] = \frac{1}{2}\left(H_P[u] + H_F[u]\right) = \overline{H_f}[u]$$

$$\overline{D_f\left(\frac{\partial p}{\partial x}\right)_f} \approx \frac{1}{2}(D_P + D_F) \times \frac{1}{2}\left(\left(\frac{\partial p}{\partial x}\right)_P + \left(\frac{\partial p}{\partial x}\right)_F\right) + O\left(\Delta x^2\right)$$

$$\overline{H_f}[u] = \frac{1}{2}\left(-u_P - D_P\left(\frac{\partial p}{\partial x}\right)_P - u_F - D_F\left(\frac{\partial p}{\partial x}\right)_F\right) = -\overline{u_f} - \overline{D_f\left(\frac{\partial p}{\partial x}\right)_f}$$

The Rhie-Chow Interpolation

$$u_f = -\overline{H_f}[u] - \overline{D_f}\left(\frac{\partial p}{\partial x}\right)_f = \underbrace{\overline{u_f}}_{\text{average velocity}} - \underbrace{\overline{D_f}\left(\left(\frac{\partial p}{\partial x}\right)_f - \overline{\left(\frac{\partial p}{\partial x}\right)_f}\right)}_{\text{correction term}}$$



$$\mathbf{v}_f = \overline{\mathbf{v}_f} - \overline{\mathbf{D}_f}\left(\nabla p_f - \overline{\nabla p_f}\right)$$

$$\dot{m}_f = \rho_f \mathbf{v}_f \cdot \mathbf{S}_f = \rho_f \overline{\mathbf{v}_f} \cdot \mathbf{S}_f - \rho_f \overline{\mathbf{D}_f}\left(\nabla p_f - \overline{\nabla p_f}\right) \cdot \mathbf{S}_f$$

$$\mathbf{v}'_f = \overline{\mathbf{v}'_f} - \overline{\mathbf{D}_f}\left(\nabla P'_f - \overline{\nabla p'_f}\right) = -\overline{\mathbf{D}_f}\nabla p'_f + \underline{\overline{\mathbf{v}'_f} + \overline{\mathbf{D}_f}\,\overline{\nabla p'_f}} = -\overline{\mathbf{D}_f}\nabla p' - \underline{\overline{\mathbf{H}_f}[\mathbf{v}']}$$

The Pressure Correction Equation

$$\sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}_f' \right) = 0$$

$$\mathbf{v}_f' = -\mathbf{D}_f \nabla (p')_f$$

$$\dot{m}_f' = \rho_f \mathbf{v}_f' \cdot \mathbf{S}_f = -\rho_f \left(\overline{\mathbf{D}} \nabla p' \right)_f \cdot \mathbf{S}_f$$

$$\sum_{f=nb(P)} \left(-\rho_f \left(\overline{\mathbf{D}} \nabla p' \right)_f \cdot \mathbf{S}_f \right) = - \sum_{f=nb(P)} \dot{m}_f^*$$

$$\dot{m}_f^* = \rho_f \mathbf{v}_f^* \cdot \mathbf{S}_f = \rho_f \overline{\mathbf{v}_f^*} \cdot \mathbf{S}_f - \rho_f \overline{\mathbf{D}_f} \left(\nabla p_f^{(n)} - \overline{\nabla p_f^{(n)}} \right) \cdot \mathbf{S}_f$$

The Pressure-Correction Equation

$$\dot{m}_f^* = \rho_f \mathbf{v}_f^* = \rho_f \overline{\mathbf{v}_f^*} \cdot \mathbf{S}_f - \rho_f^{(n)} \overline{\mathbf{D}}_f \left(\nabla p_f^{(n)} - \overline{\nabla p_f^{(n)}} \right) \cdot \mathbf{S}_f$$

$$a_P p'_P + \sum_{F=nb(P)} a_F p'_F = b_P$$

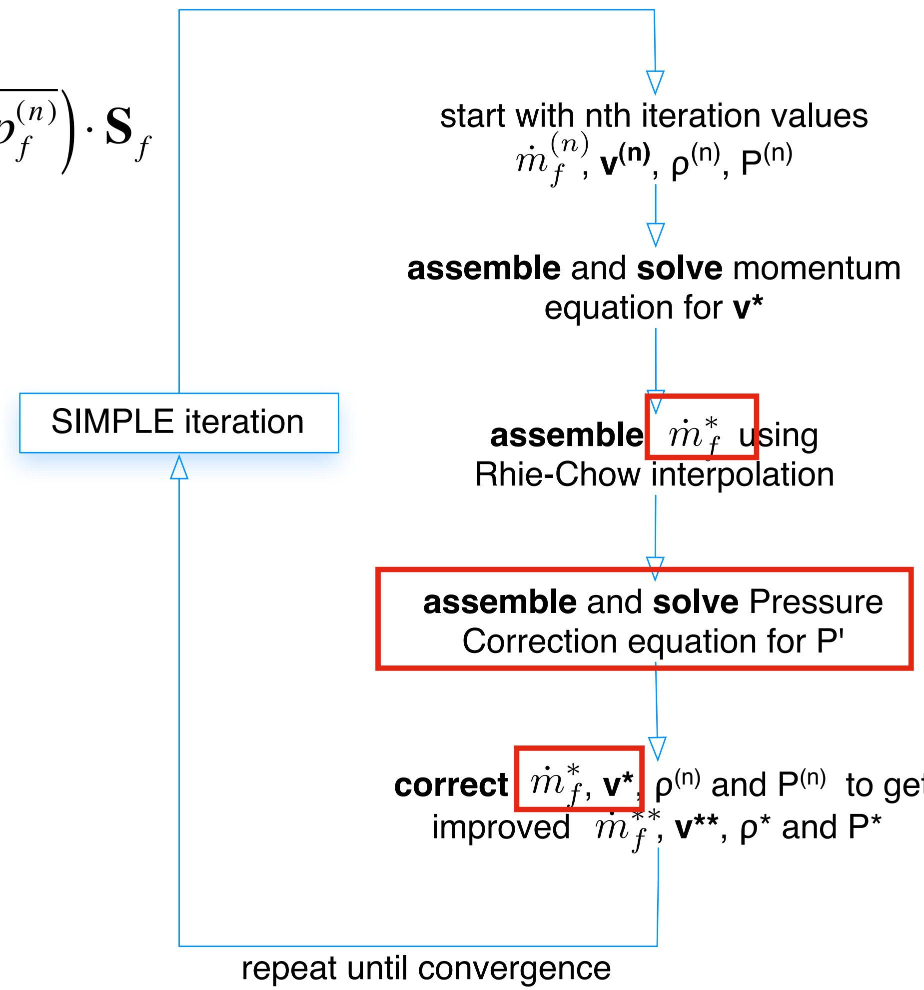
$$b_P = -\sum_{f=nb(P)} \dot{m}_f^* + \sum_{f=nb(P)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p'_f \right) \cdot \mathbf{T}_f$$

$$a_F = -\rho_f^{(n)} \mathcal{D}_f$$

$$a_P = \sum_{f=nb(P)} \rho_f^{(n)} \mathcal{D}_f$$

$$\dot{m}'_f = -\rho_f^{(n)} \left(\overline{\mathbf{D}}_f \nabla p'_f \right) \cdot \mathbf{S}_f$$

$$\mathbf{v}'_P = -\mathbf{D}_P \nabla p'_P$$



Compressible Flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot \tau - \nabla p + \mathbf{B}$$

$$\rho = C_\rho p$$

$$\rho\big|_{(P^* + P')} = \rho\big|_{(P)} + \frac{\partial \rho}{\partial P} p' \Rightarrow \rho' = \left(\frac{\partial \rho}{\partial P}\right) p' = C_\rho p'$$

$$p = p^{(n)} + p'$$

$$\rho = \rho^{(n)} + \rho'$$

$$\mathbf{v} = \mathbf{v}^* + \mathbf{v}'$$

Discretized Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Incompressible

$$\sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}_f' \right) = 0$$

Compressible

$$\frac{\left(\rho_P + \rho_P' - \rho_P^{(n)} \right)}{\Delta t} V_P + \sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}_f' \right) = 0$$

$$\dot{m}_f = \rho_f \left(\mathbf{v}_f^* + \mathbf{v}_f' \right) \cdot \mathbf{S}_f = \underbrace{\rho_f \mathbf{v}_f^* \cdot \mathbf{S}_f}_{\dot{m}_f^*} + \underbrace{\rho_f \mathbf{v}_f' \cdot \mathbf{S}_f}_{\dot{m}_f'}$$

$$\dot{m}_f = \left(\rho_f^{(n)} + \rho_f' \right) \left(\mathbf{v}_f^* + \mathbf{v}_f' \right) \cdot \mathbf{S}_f$$

$$= \underbrace{\rho_f^{(n)} \mathbf{v}_f^* \cdot \mathbf{S}_f}_{\dot{m}_f^*} + \underbrace{\rho_f^{(n)} \mathbf{v}_f' \cdot \mathbf{S}_f + \rho_f' \mathbf{v}_f^* \cdot \mathbf{S}_f + \rho_f' \mathbf{v}_f' \cdot \mathbf{S}_f}_{\dot{m}_f'}$$

Velocity Correction

$$\boxed{\underbrace{\rho_f^{(n)}\mathbf{v}_f^{*}\cdot\mathbf{S}_f}_{\dot{m}_f^{*}}}+\boxed{\underbrace{\rho_f^{(n)}\mathbf{v}'_f\cdot\mathbf{S}_f+\rho'_f\mathbf{v}_f^{*}\cdot\mathbf{S}_f+\rho'_f\mathbf{v}'_f\cdot\mathbf{S}_f}_{\dot{m}'_f}}$$

$$\Rightarrow \qquad \dot{m}_f^{*}=\rho_f^{(n)}\overline{\mathbf{v}_f^{*}}\cdot\mathbf{S}_f-\rho_f^{(n)}\overline{\mathbf{D}_f}\left(\nabla p_f^{(n)}-\overline{\nabla p_f^{(n)}}\right)\cdot\mathbf{S}_f$$

$$\Rightarrow \qquad \dot{m}'_f=\underbrace{\rho_f^{(n)}\overline{\mathbf{v}'_f}\cdot\mathbf{S}_f-\rho_f^{(n)}\overline{\mathbf{D}_f}\left(\nabla p'_f-\overline{\nabla p'_f}\right)\cdot\mathbf{S}_f}_{(1)}+\underbrace{\left(\frac{\dot{m}_f^{*}}{\rho_f^{(n)}}\cdot\mathbf{S}_f\right)C_{\rho,f}p'_f}_{(2)}$$

Pressure Equation

$$\boxed{\frac{\left(\rho_P + \rho'_P - \rho_P^{(n)}\right)}{\Delta t} V_P + \sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}'_f\right) = 0}$$

$$\frac{V_P}{\Delta t} C_\rho P'_P + \sum_{f=nb(P)} \left\{ -\rho_f^{(n)} \overline{\mathbf{D}_f} \nabla p'_f \cdot \mathbf{S}_f + \left(\frac{\dot{m}_f^*}{\rho_f^{(n)}}\right) C_\rho p'_f \right\} = -\left(\frac{\rho_P^{(n)} - \rho_P^o}{\Delta t} V_P + \sum_{f=nb(P)} \dot{m}_f^*\right) - \underline{\sum_{f=nb(P)} \rho_f^{(n)} \left(\overline{\mathbf{v}'_f} + \overline{\mathbf{D}_f} \overline{\nabla p'_f}\right) \cdot \mathbf{S}_f} - \underline{\underline{\sum_{f=nb(P)} \rho'_f \mathbf{v}'_f \cdot \mathbf{S}_f}}$$

$$\sum_{f=nb(P)} \left(\rho'_f \overline{\mathbf{v}'_f} + \overline{\mathbf{D}_f} \overline{\nabla p'_f}\right) \cdot \mathbf{S}_f = \sum_{f=nb(P)} -\overline{\mathbf{H}_f} [\mathbf{v}'] \cdot \mathbf{S}_f = \sum_{f=nb(P)} -0.5 \left(\mathbf{H}'_P + \mathbf{H}'_N\right) \cdot \mathbf{S}_f$$

$$= \sum_{f=nb(P)} -0.5 \left(\sum_{NBP(P)} \left(\frac{a_{NBP}}{a_P} \mathbf{v}'_{NBP}\right) + \sum_{NB(F)} \left(\frac{a_{NBF}}{a_F} \mathbf{v}'_{NBF}\right)\right) \cdot \mathbf{S}_f$$

Pressure Equation

$$\underbrace{\frac{V_P C_\rho}{\Delta t} p'_P}_{\substack{transient-like\\ term}} + \underbrace{\sum_{f=nb(P)} \left[C_\rho \left(\frac{\dot{m}_f^*}{\rho_f^{(n)}} \right) p'_f \right]}_{convection-like\ term} + \underbrace{\sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathbf{D}_f \left(\nabla p' \right)_f \cdot \mathbf{S}_f \right]}_{diffusion-like\ term} = \underbrace{-\left(\frac{\left(\rho_P^{(n)} - \rho_P^{\circ} \right)}{\Delta t} V_P + \sum_{f=nb(P)} \dot{m}_f^* \right)}_{source-like\ term}$$

$$\sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathbf{D}_f \left(\nabla p' \right)_f \cdot \mathbf{S}_f \right] = \sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathbf{D}_f \left(\nabla p' \right)_f \cdot \left(\mathbf{E}_f + \mathbf{T}_f \right) \right] = \sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathsf{D}_\mathsf{f} \left(p'_F - p'_P \right) \right]$$

$$b_P = -\left(\frac{\left(\rho_P^{(n)} - \rho_P^{\circ} \right)}{\Delta t} V_P + \sum_{f=nb(P)} \dot{m}_f^* \right) + \sum_{f=nb(P)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p'_f \right) \cdot \mathbf{T}_f$$

$$\mathcal{D}_\mathsf{f} = \left(\frac{\overline{d_f^u} E_{x,f} + \overline{d_f^v} E_{y,f}}{\left\| \mathbf{d}_{PF} \right\|} \right)$$

$$a_F = -\left\| -\dot{m}_f^*, 0 \right\| \frac{C_{\rho,f}}{\rho_f^{(n)}} - \rho_f^{(n)} \mathcal{D}_f$$

$$a_P = \frac{V_P C_\rho}{\Delta t} + \sum_{f=nb(P)} \left(\frac{C_{\rho,f}}{\rho_f^{(n)}} \left\| \dot{m}_f^*, 0 \right\| \right) + \sum_{f=nb(P)} \rho_f^{(n)} \mathcal{D}_f$$