The efficiency at maximum power output of endoreversible engines under combined heat transfer modes

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Received 20th May 1994
Revised 11th October 1994

A mathematical model for investigating the performance of endoreversible heat engines under combined conduction, convection, and radiation heat transfer modes is presented. The model is suitable to be introduced to engineering students attending a course in thermodynamics who may apply it to predicting the performance of real engines and a variety of energy conversion systems in a simplified manner. Results generated by the model show that the relative contribution of conduction/convection and radiation heat transfer modes deeply affect the efficiency at maximum power output. Moreover, a number of well-known formulae presented in several references are shown to represent special cases of the new formulation.

NOMENCLATURE

a_1, a_2	coefficients in algebraic equation
b_1, b_2	coefficients in algebraic equation
c_1, c_2	coefficients in algebraic equation
d_2	coefficient in algebraic equation
e_2	coefficient in algebraic equation
Q_1	heat entering the Carnot engine
Q_2	heat leaving the Carnot engine
R_1, R_4	dimensionless parameters
S_1, S_4	dimensionless parameters
t_{01}	dimensional temperature of the hot reservoir (i.e. the working material entering the engine), K
t_{02}	dimensional temperature of the cold reservoir (i.e. the working material leaving the engine), K

T_1	dimensional temperature of the heat source, K
t_1	dimensionless temperature of the hot reservoir (i.e. the working material entering the engine)
T_2	dimensional temperature of the heat sink, K
V_1, V_4	dimensionless parameters
W	power output
W_m	maximum power output
α_1, α_4	heat transfer coefficients
β_1, β_4	heat transfer coefficients
γ_1, γ_4	heat transfer coefficients
η	efficiency
η_m	efficiency at maximum power
τ	ratio of heat sink to heat source temperatures

INTRODUCTION

The use of the Carnot concept in classrooms is usually confined to setting an upper bound on the efficiencies that can be achieved by actual engines. By exploiting endoreversible or finite-time thermodynamics, this limited usefulness of the theoretical Carnot engine may be greatly expanded to studying the performance of real engines and a variety of interesting energy conversion systems by treating them as Carnot-like engines. By modelling real engines as endoreversible engines all irreversibilities are assumed to occur between the working material and the heat reservoirs while energy conversion within the engine itself is considered to take place reversibly. The use of the concept of endoreversibility, allows the simplification of complicated thermal systems and makes them easier for students to understand. However, it should be made clear that the method does not allow detailed analysis of the problem considered and rather, a simplified view of the system performance can only be obtained. Nevertheless, this simplified view may be of great value for a designer trying to choose among several suggested systems before going into detailed analysis.

The work in the subject was initiated by Curzon and Ahlborn [1] who employed Newton's law of cooling to describe the heat fluxes across the walls of the hot and cold reservoirs. Chen and Yan [2], generalized the work reported in Reference 1 by assuming the rate of heat flowing through the walls of the reservoirs to be ruled by an equation of the form:

$$Q = \alpha\left(T_1^n - T_2^n\right) \tag{1}$$

where n is a non-zero integer. DeVos [3], simplified the analysis presented in References 1 and 2 by developing a simpler model for studying the engine performance.

Several workers [4–7] have also used endoreversible thermodynamics for predicting a variety of phenomena. Gordon [4], applied finite-time thermodynamics to analyse the thermoelectric generator. Gordon and Zarmi [5] and DeVos and Flater [6], modelled the earth and its envelope using a Carnot-like engine with its heat input being solar radiation and its work output representing the wind generated. Nuwayhid and Moukalled [7], added a heat leak term into the model of DeVos and Flater [6] and studied the effect of a planet thermal conductance on conversion efficiency of solar energy into wind energy. Nulton *et al.* [8] and Pathria *et al.* [9] described a set of feasible operations of a finite-time heat engine subject only to thermal losses in terms of an inequality similar to the second law of thermodynamics and applied it to Carnot-like refrigerators and heat pumps. Recently, Moukalled and

Nuwayhid [10] expanded the applicability and usefulness of the Curzon–Ahlborn concept and made it more realistic by adding a heat leak term into a variation of the DeVos model [3]. Their work, however, was valid for the case when the operating temperatures of the engine were not too high (i.e. dominant conduction/convection and negligible radiation heat transfer modes).

It is the intention of this work to remove the shortcomings of the model developed by Moukalled and Nuwayhid [10] and to extend it into situations where radiation heat transfer is as important as conduction and convection. As will be seen, this combined mode of heat transfer results in very complicated algebraic equations which, in general, have to be tackled numerically. Furthermore, results reported in References 1, 2 and 10 are shown to represent special cases of the new formulation.

THE COMBINED CONDUCTION, CONVECTION, AND RADIATION MODEL

A schematic of the Carnot-like engine under consideration is depicted in Fig. 1. As shown, heat exchange occurs via combined conduction, convection, and radiation heat transfer modes. Furthermore, heat transfer between the heat source (or sink) and the hot (or cold) reservoir of the engine takes place irreversibly, while heat exchange from the hot reservoir to the engine and from the engine to the cold reservoir occurs reversibly. The external heat loss from the engine is modelled via the heat-leak term. As discussed by Kiang and Wu [11], two different approaches have been used in analysing endoreversible Carnot cycles. These two methods have been denoted by the Curzon–Ahlborn and the Bejan approaches. In the Curzon–Ahlborn method, the Carnot engine is treated as a reciprocating engine. In the Bejan approach however, the Carnot engine represents a steady-flow engine. In this work, the Bejan approach is adopted [11]. Referring to the model shown in Fig. 1, the heat transfer from the hot reservoir into the engine Q_1 is given by a heat balance equation as

$$Q_1 = \alpha_1 (T_1 - t_{01}) + \alpha_4 \left(T_1^4 - t_{01}^4\right) - \gamma_1 (t_{01} - t_{02}) - \gamma_4 \left(t_{01}^2 - t_{02}^4\right) \tag{2}$$

while the heat transfer from the engine to the cold reservoir is

$$Q_2 = \beta_1 (t_{02} - T_2) + \beta_4 \left(t_{02}^4 - T_2^4\right) - \gamma_1 (t_{01} - t_{02}) - \gamma_4 \left(t_{01}^4 - t_{02}^4\right) \tag{3}$$

where α_1, α_4, β_1, β_4, γ_1, and γ_4 are the conduction/convection and radiation heat transfer coefficients. In the above equations, Q_1 and Q_2 are defined as positive quantities. Endoreversibility for the engine requires that

$$\frac{Q_1}{t_{01}} = \frac{Q_2}{t_{02}} \tag{4}$$

with the Carnot efficiency given by

$$\eta = 1 - \frac{t_{02}}{t_{01}} \tag{5}$$

The work can therefore be found from

$$W = \alpha_1 (T_1 - t_{01}) + \alpha_4 \left(T_1^4 - t_{01}^4\right) - \beta_1 (t_{02} - T_2) - \beta_4 \left(t_{02}^4 - T_2^4\right) \tag{6}$$

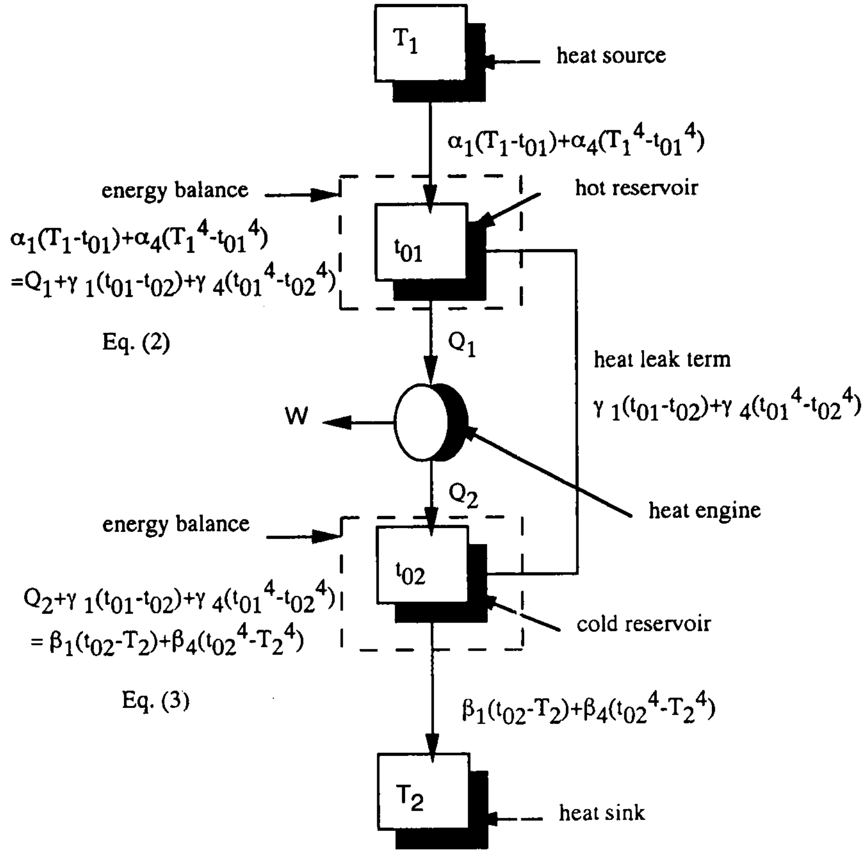

Fig. 1. Schematic of an endoreversible engine with heat leak operating between heat source at T_1 and heat sink at T_2.

In order to reduce the number of parameters involved and to widen the applicability of results the following dimensionless quantities are defined:

$$V_1 = \frac{\alpha_1 T_1}{\alpha_1 T_1 + \alpha_4 T_1^4}, \quad V_4 = \frac{\alpha_4 T_1^4}{\alpha_1 T_1 + \alpha_4 T_1^4}, \quad V_1 + V_4 = 1 \tag{7a}$$

$$R_1 = \frac{\beta_1 T_1}{\alpha_1 T_1 + \alpha_4 T_1^4}, \quad R_4 = \frac{\beta_4 T_1^4}{\alpha_1 T_1 + \alpha_4 T_1^4} \tag{7b}$$

$$S_1 = \frac{\gamma_1 T_1}{\alpha_1 T_1 + \alpha_4 T_1^4}, \quad S_4 = \frac{\gamma_4 T_1^4}{\alpha_1 T_1 + \alpha_4 T_1^4} \tag{7c}$$

$$\tau = \frac{T_2}{T_1}, \quad t_1 = \frac{t_{01}}{T_1} \tag{7d}$$

where V_1, V_4, R_1, R_4, S_1, S_4 are dimensionless heat transfer coefficients, τ is the ratio of the temperature of the heat sink to that of the heat source, and t_1 is the dimensionless counterpart of the hot reservoir temperature. Applying the reversibility condition by inserting Q_1 and Q_2 into equation (4), and using the Carnot efficiency (equation (5)), the following equations for t_1 and W are obtained:

$$a_1 t_1^4 + b_1 t_1 + c_1 = 0 \tag{8a}$$

where

$$a_1 = \eta^5 S_4 + \eta^4 (R_4 - 4S_4) + \eta^3 (-4R_4 + 6S_4) + \eta^2 (6R_4 - 4S_4) - \eta(4R_4 + V_4) + R_4 + V_4 \tag{8b}$$

$$b_1 = -\eta^2 S_1 - \eta(R_1 + V_1) + R_1 + V_1 \tag{8c}$$

$$c_1 = \eta - R_4\tau^4 - R_1\tau - 1 \tag{8d}$$

and

$$\frac{W}{\alpha_1 T_1 + \alpha_4 T_1^4} = 1 + R_1\tau + R_4\tau^4 - V_1 t_1 - V_4 t_1^4 - R_1(1-\eta)t_1 - R_4(1-\eta)^4 t_1^4 \tag{9}$$

Equations (8) and (9) are used to obtain the efficiency of the endoreversible heat engine at maximum power. For this purpose, the derivative of the normalized power equation (equation (9)) with respect to the efficiency is set to zero. This results in the following relation:

$$\frac{dt_1}{d\eta} = \frac{R_1 t_1 + 4(1-\eta)^3 R_4 t_1^4}{(1-\eta)R_1 + 4(1-\eta)^4 R_4 t_1^3 + V_1 + 4t_1^3 V_4} \tag{10}$$

A second equation for the derivative of t_1 with respect to η may be obtained from the reversibility equation, equation (8), and is given by

$$\frac{dt_1}{d\eta} = \frac{-1 + t_1(R_1 + 2\eta S_1 + V_1) + t_1^4 \begin{bmatrix} 4R_4 + \eta^2(12R_4 - 18S_4) - 5\eta^4 S_4 \\ +\eta(-12R_4 + 8S_4) + \eta^3(-4R_4 + 16S_4) + V_4 \end{bmatrix}}{D} \tag{11a}$$

where

$$D = R_1 - \eta^2 S_1 - \eta(R_1 + V_1) + V_1 + t_1^3 \big[4R_4 + \eta^4(4R_4 - 16S_4) + \eta^2(24R_4 - 16S_4) + 4\eta^5 S_4 + \eta^3(-16R_4 + 24S_4) - \eta(16R_4 + 4V_4) + 4V_4\big] \tag{11b}$$

Equating equations (10) and (11), a relation for the efficiency at maximum power (η_m) is obtained and its final form is written as

$$a_2 t_1^7 + b_2 t_1^4 + c_2 t_1^3 + d_2 t_1 + e_2 = 0 \tag{12a}$$

where

$$a_2 = \left[-32\eta_m + 136\eta_m^2 - 256\eta_m^3 + 292\eta_m^4 - 224\eta_m^5 + 112\eta_m^6 - 32\eta_m^7 + 4\eta_m^8\right]R_4S_4 + \left[-4 + 24\eta_m^2 - 32\eta_m^3 + 12\eta_m^4\right]R_4V_4 + \left[-32\eta_m + 72\eta_m^2 - 64\eta_m^3 + 20\eta_m^4\right]S_4V_4 - 4V_4^2 \tag{12b}$$

$$b_2 = \left[-8\eta_m + 28\eta_m^2 - 36\eta_m^3 + 20\eta_m^4 - 4\eta_m^5\right]R_4S_1 + \left[-8\eta_m + 10\eta_m^2 - 10\eta_m^3 + 5\eta_m^4 - \eta_m^5\right]R_1S_4 + \left[-4 + 12\eta_m - 12\eta_m^2 + 4\eta_m^3\right]R_4V_1 + \left[-8\eta_m + 18\eta_m^2 - 16\eta_m^3 + 5\eta_m^4\right]S_4V_1 - (3\eta_m + 1)R_1V_4 - 8\eta_m S_1V_4 - 5V_1V_4 \tag{12c}$$

$$c_2 = \left[4 - 16\eta_m + 24\eta_m^2 - 16\eta_m^3 + 4\eta_m^4\right]R_4 + 4V_4 \tag{12d}$$

$$d_2 = \left[-2\eta_m + \eta_m^2\right]R_1S_1 - 2\eta_m S_1V_1 - R_1V_1 - V_1^2 \tag{12e}$$

$$e_2 = (1 - \eta_m)R_1 + V_1 \tag{12f}$$

At the same time, the efficiency should satisfy the reversibility equation (equation (8)). This results in a highly nonlinear system of two equations in the two unknowns t_1 and η_m. Therefore, the problem is mathematically well defined and in general, the solution may be obtained numerically once the constant parameters are assigned specific values. Such a numerical solution is given here for several combinations of the parameters involved and results are displayed graphically in Figs 2–4.

The conversion efficiency at maximum power, as given by the above equations, is a function of the ratio of the cold to hot reservoir temperatures τ, the relative contribution of conduction/convection and radiation to total heat transfer between the working fluid and the heat source (V_1 and V_4), the dimensionless heat transfer coefficients between the working fluid and the heat sink (R_1 and R_4), and the dimensionless heat leak coefficients (S_1 and S_4). In Figs 2–4, η_m is plotted as a function of R_1 and R_4 for different values of V_1, V_4, S_1, and S_4 at a given τ. The general trend of results is similar and shows η_m, for constant values of τ, to increase with R_1 and R_4 for given V_1, V_4, S_1, and S_4 and to decrease with increasing τ for given values of the various parameters involved. This is to be expected since, when S_1, S_4, and τ are constant, increasing R_1 and R_4 (the dimensionless heat transfer coefficients) reflect an increase in the heat transfer coefficients or a decrease in resistance to heat flow between the working fluid and the heat sink and results in a lower temperature for the cold reservoir. This, in turn, causes the Carnot-like engine to operate between a hot and cold reservoirs of higher temperature difference and consequently, results in an increase of its efficiency. Furthermore, at constant values of R_1, R_4, S_1 and S_4, an increase in τ produces closer hot and cold reservoir temperatures and hence a less efficient engine.

By comparing results in Figs 2–4, it can easily be inferred that, the efficiency at maximum power conditions of the endoreversible engine at given R_1 and R_4 increases with increasing the dimensionless radiation heat transfer coefficients, i.e. with V_4 and/or S_4. This is so because as V_4 and S_4 increase, the contribution of radiation heat transfer increases

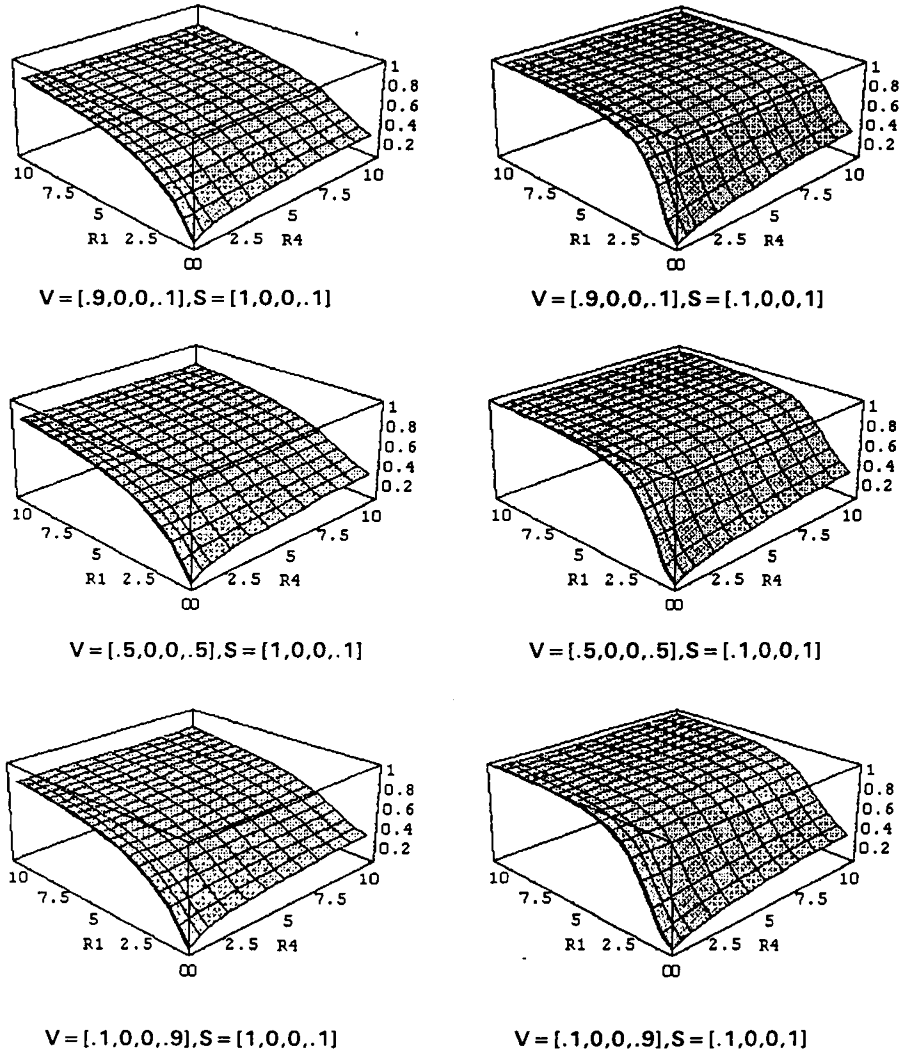

Fig. 2. Variation of the efficiency at maximum power with R_1 and R_4 for $T_2/T_1 = 0$.

implying higher hot reservoir temperature and thereby higher efficiency. Results in Fig. 2 ($\tau = 0$) are of theoretical importance because they show the highest possible efficiencies of an endoreversible Carnot engine.

DERIVATION OF PREVIOUSLY REPORTED WORK FROM THE NEW MODEL

In this section, the work reported in several references is shown to represent special cases of the new formulation presented here. This is done by setting alternatively the radiation and conduction/convection heat transfer coefficients to zero.

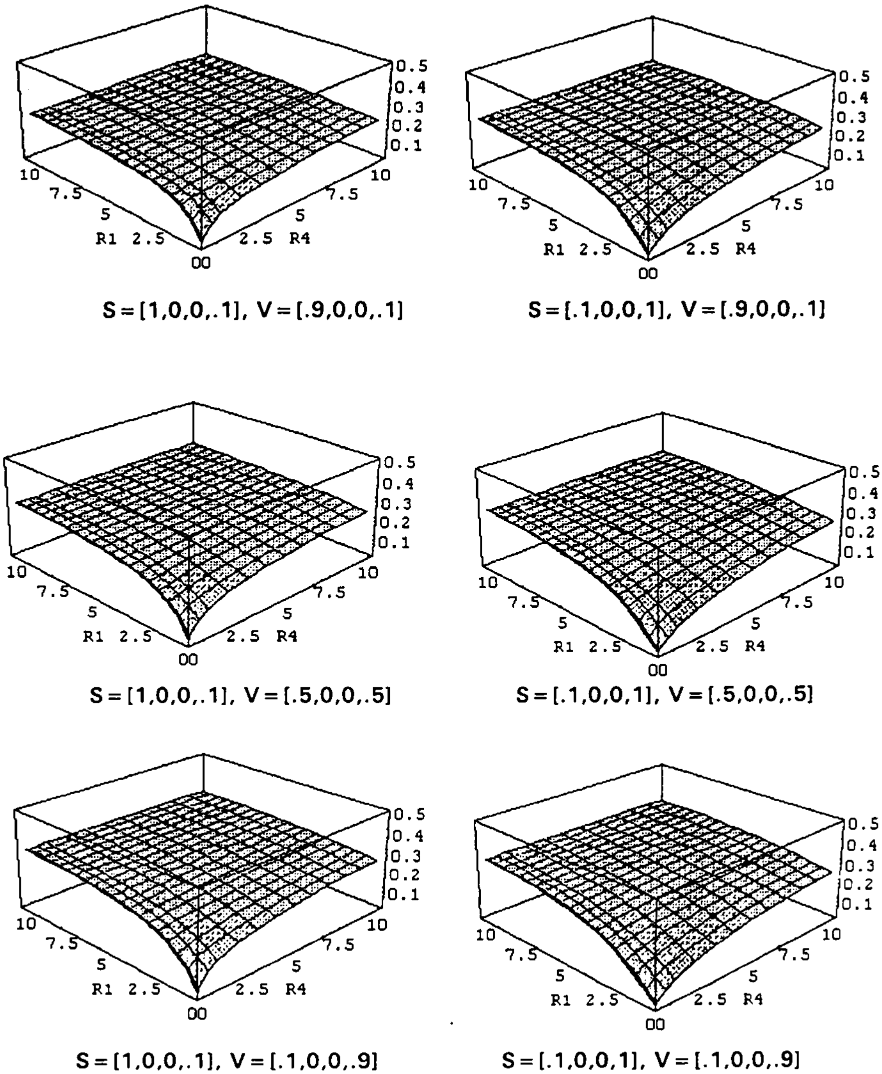

Fig. 3. Variation of the efficiency at maximum power with R_1 and R_4 for $T_2/T_1 = 0.4$.

Case 1: Dominant conduction/convection heat transfer mode

In this case, all heat transfer processes are governed by Newton's law of cooling and the relations for t_1 and W should be the same as those presented by Moukalled and Nuwayhid [10] and may be obtained from equations (8) and (9) by setting $V_4 = R_4 = S_4 = 0$, $V_1 = 1$, $R_1 = R$, and $S_1 = S$. Performing this step, the following equations for t_1 and W are obtained:

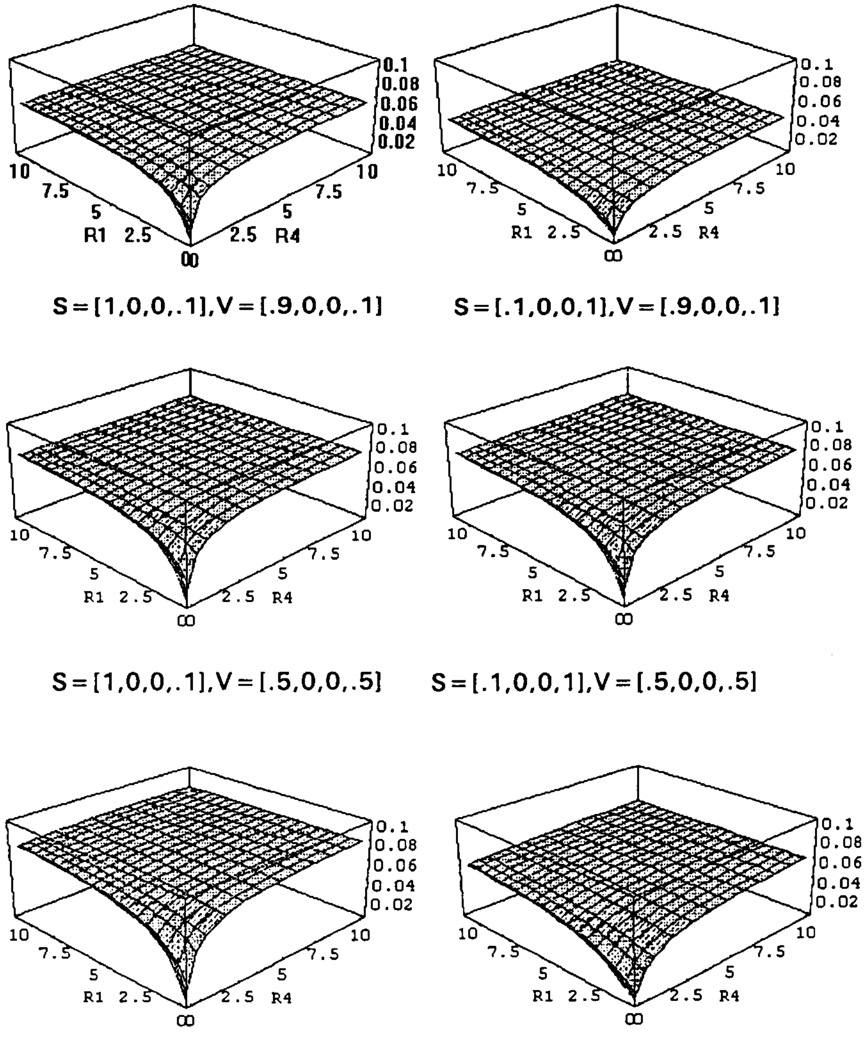

Fig. 4. Variation of the efficiency at maximum power with R_1 and R_4 for $T_2/T_1 = 0.8$.

$$t_1 = \frac{1-\eta+R\tau}{(1+R)(1-\eta)-S\eta^2} \tag{13}$$

$$\frac{W}{\alpha T_1} = 1+R\tau - \frac{\left[(1-\eta)+R\tau\right]\left[1+R(1-\eta)\right]}{(1+R)(1-\eta)-S\eta^2} \tag{14}$$

The efficiency at maximum power is found by differentiating the power equation (equation (14)) with respect to η and setting the result to zero. This results in the following equation for the efficiency at maximum power (η_m):

$$\left[R^2(1+S\tau)+R(1+2S)+S\right]\eta_m^2-2\left[R^2(1+S\tau)+R(1+S+S\tau)+S\right]\eta_m+R(1+R)(1-\tau)=0 \tag{15}$$

The above equation for η_m has two real roots with the one physically correct (i.e. positive) given by

$$\eta_m=1-\frac{(1-\tau)RS}{R^2(1+\tau S)+R(1+2S)+S}-\sqrt{\left[1-\frac{(1-\tau)RS}{R^2(1+\tau S)+R(1+2S)+S}\right]^2-\frac{R(1+R)(1-\tau)}{R^2(1+\tau S)+R(1+2S)+S}} \tag{16}$$

It is interesting to note that when there is no leakage ($S=0$), equation (16) above, irrespective of the value of R, reduces to

$$\eta_m=1-\sqrt{\tau} \tag{17}$$

which is the Curzon–Ahlborn efficiency [1]. The maximum work under these conditions is obtained from the following simple equation:

$$\frac{W_m}{\alpha T_1}=\frac{R}{R+1}\eta_m^2 \tag{18}$$

Therefore equations (14) (with W replaced by W_m and η by η_m) and (16) are the generalized forms of equations (18) and (17), respectively.

Case 2: Dominant radiation heat transfer mode

For this cases, all heat transfer processes including the heat leak, take place through a radiative heat transfer mode. Upon setting $V_1=R_1=S_1=0$, $V_4=1$, $R_4=R$, and $S_4=S$, the following relations for t_1 and η_m are obtained:

$$t_1=\left[\frac{(1-\eta_m)+\tau^4R}{(R+\eta_mS)(1-\eta_m)^4+(1-\eta_m)-\eta_mS}\right]^{1/4} \tag{19}$$

and

$$\begin{aligned}&R\left[S+R+\tau^4SR\right](1-\eta_m)^8+4[RS+R+S](1-\eta_m)^5\\&-\left[3S+5RS+3R-3SR^2\tau^4-5RS\tau^4-3R^2\tau^4\right](1-\eta_m)^4\\&-4R[RS+R+S]\tau^4(1-\eta_m)^3-S-R\tau^4-RS\tau^4=0\end{aligned} \tag{20}$$

The highest possible efficiency is attained when there is no leakage ($S = 0$) and when $\tau = 0$. If this is the case, then equation (20) may be written as

$$R(1-\eta_m)^4 + 4(1-\eta_m) - 3 = 0 \tag{21}$$

This equation reduces to that of DeVos [6] when $R \to 1$, i.e.,

$$\eta_m^4 - 4\eta_m^3 + 6\eta_m^2 - 8\eta_m + 2 = 0 \tag{22}$$

and the solution gives $\eta_m = 0.307$. Equation (21), is however, the more general one in that the variation of efficiency at maximum power with R is shown.

Finally, of interest is the case for which $S = 0$ and $R \to \infty$ (i.e. when the only thermal resistance is between the working fluid and the high temperature source). Substitution of these values into equations (19) and (20) results in the following equations for the efficiency at maximum power and t_{01}:

$$\eta_m = 1 - \frac{T_2}{t_{01}} \tag{23}$$

and

$$4t_{01}^5 - 3t_{01}^4 T_2 - T_1^4 T_2 = 0 \tag{24}$$

The last equation, known as Castan's relation [2], is a practical formula in solar energy conversion systems and shows that the results of this paper are more general.

CONCLUSION

A theoretical investigation of the performance of Carnot-like engines with heat leak was undertaken. The heat exchange processes were assumed to occur via combined conduction, convection, and radiation heat transfer modes. Generated results, demonstrated the strong influence of the heat transfer mode on the efficiency at maximum power. Several well-established formulae were shown to represent special cases of the new formulation. The simplicity of the approach (one has to solve algebraic equations only which is a trivial numerical task), makes it an attractive tool to students, who may employ it for predicting the performance of a variety of energy conversion systems (e.g. power plants [12], thermoelectric generators [4], and solar energy conversion systems [7]).

ACKNOWLEDGEMENT

The financial support provided by the University Research Board of the American University of Beirut through Grant No. 113040-48816 is gratefully acknowledged.

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