The efficiency at maximum power output of a Carnot engine with heat leak

F. Moukalled and **R. Y. Nuwayhid**, Faculty of Engineering & Architecture, Mechanical Engineering Department, American University of Beirut, Beirut, Lebanon.

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Endoreversible thermodynamics are used for studying the performance of Carnot engines with heat leak. This is done by adding a heat leak term into a variation of the model suggested by DeVos [1]. Heat transfer across the engine is assumed to occur via a conduction/convection mechanism and Newton's law of cooling is employed to model the heat transfer processes. The efficiency at maximum power output is found to be deeply affected by the rate of heat leak. Moreover, the Curzon–Ahlborn relation [2] is shown to represent a special case of the new formulation. Since the suggested model allows more flexibility in predicting actual engines' performance, its use is recommended in thermodynamics courses.

NOMENCLATURE

Q_1	heat entering the Carnot engine
Q_2	heat leaving the Carnot engine
R	dimensionless parameter
S	dimensionless parameter
t_{01}	dimensional temperature of the hot reservoir (i.e. the working material entering the engine), K
t_{02}	dimensional temperature of the cold reservoir (i.e. the working material leaving the engine), K
T_1	dimensional temperature of the heat source, K
t_1	dimensionless temperature of the hot reservoir (i.e. the working material entering the engine)
T_2	dimensional temperature of the heat sink, K
W	power output
W_m	maximum power output
α, β, γ	heat transfer coefficients
η	efficiency
η_m	efficiency at maximum power
τ	ratio of heat sink to heat source temperatures

INTRODUCTION

In classrooms, the Carnot cycle is generally introduced as being the most efficient cycle operating between two thermal reservoirs. Therefore, the Carnot concept sets an upper limit on the efficiencies that can be achieved by actual cycles operating between two reservoirs of fixed temperature. This limited use of the Carnot engine was removed through the introduction of what is called finite-time or endoreversible thermodynamics. The basic concept of an endoreversible heat engine is to consider the irreversibility associated with heat transfer between the working material and the heat reservoirs while processes within the engine are assumed to take place ideally. With this concept, very complicated thermal systems may be simplified and made easier for students to understand. However, it should be pointed out that the approach does not allow detailed predictions of the problem considered and, rather, a simplified view of the system performance only can be obtained. Nevertheless, this simplified view may be of great value for a designer trying to choose among several suggested systems before going into detailed analysis.

Curzon and Ahlborn [2] exploited the concept of finite-time thermodynamics for predicting the performance of real engines by treating them as Carnot-like engines. In their work, the heat fluxes across the walls of the hot and cold reservoirs were taken to be proportional to the prevailing temperature differences there (i.e. Newton's law of cooling). Chen and Yan [3], generalized the work reported in reference 2 by assuming the rate of heat flowing through the walls of the reservoirs to be ruled by an equation of the form:

$$Q = \alpha(T_1^n - T_2^n) \tag{1}$$

where n is a nonzero integer. DeVos [1], simplified the analysis presented in references 1 and 2 by developing a simpler model for studying the engine performance.

Several workers [4–7] have also used endoreversible thermodynamics for predicting a variety of interesting phenomena. Gordon [4], applied finite-time thermodynamics to analyse a thermoelectric generator. Gordon and Zarmi [5] and DeVos and Flater [6], modelled the Earth and its envelope using a Carnot-like engine with its heat input being solar radiation and its work output representing the wind generated. Nuwayhid and Moukalled [7], added a heat leak term into the model of DeVos and Flater [6] and studied the effect of a planet thermal conductance on conversion efficiency of solar energy into wind energy. The theoretical upper bound on conversion efficiency reported in reference 6 was shown to be well above the actual values predicted by the modified model. Recently, Nulton *et al.* [8] and Pathria *et al.* [9] described a set of feasible operations of a finite-time heat engine subject only to thermal losses in terms of an inequality similar to the second law of thermodynamics and applied it to Carnot-like refrigerators and heat pumps.

From the above literature survey it appears that, even though heat leak has a realistic influence on model performance [7], it has not been widely exploited. Adding a heat-leak term into the model, makes the model more realistic since, the heat-leak term serves as a means to account for the external thermal losses from the engine to the surroundings. Therefore, the intention of this work is to further expand the applicability and usefulness of the Curzon–Ahlborn's concept and to make it more realistic by adding a heat-leak term into a variation of DeVos model [1]. In addition, all heat fluxes are assumed to be proportional to the prevailing temperature difference across their respective walls. This is equivalent to assuming dominant conduction/convection heat transfer modes. Such an assumption is valid if the operating temperatures of the engine are such that the relative contribution of radiation heat transfer is negligible.

THE HEAT-LEAK MODEL

A schematic of the Carnot-like engine under consideration is depicted in Fig. 1. Heat transfer between the heat source (or sink) and the hot (or cold) reservoir of the engine takes place irreversibly, while heat transfer from the hot reservoir to the engine and from the engine to the cold reservoir occurs reversibly. The external thermal losses from the engine are modelled via the heat-leak term. With such a model (Fig. 1), the heat transfer from the hot reservoir into the engine Q_1 is given by an energy balance on the hot reservoir as

$$Q_1 = \alpha(T_1 - t_{01}) - \gamma(t_{01} - t_{02}) \tag{2}$$

while the heat transfer from the engine to the cold reservoir is obtained from an energy balance on the cold reservoir:

$$Q_2 = \beta(t_{01} - T_2) - \gamma(t_{01} - t_{02}) \tag{3}$$

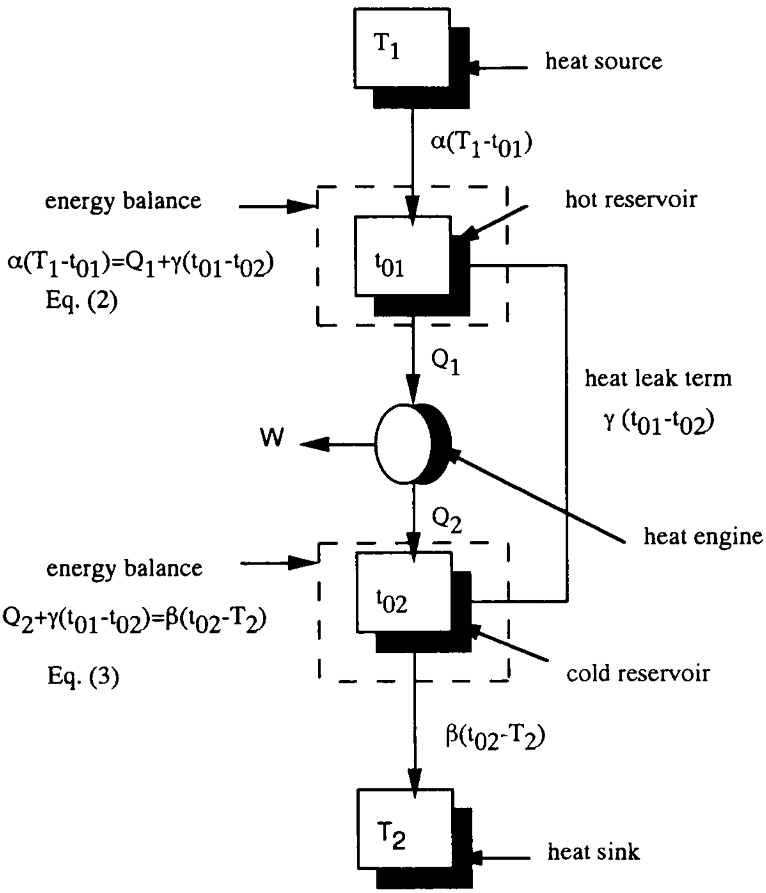

Fig. 1. Schematic of an endoreversible engine with heat leak operating between heat source at T_1 and heat sink at T_2.

where α, β, and γ are the conduction/convection heat transfer coefficients. In the above equations, Q_1 and Q_2 are defined as positive quantities. Endoreversibility requires that

$$\frac{Q_1}{t_{01}} = \frac{Q_2}{t_{02}} \tag{4}$$

with the Carnot efficiency given by

$$\eta = 1 - \frac{t_{02}}{t_{01}} \tag{5}$$

The work can therefore be found from

$$W = \alpha(T_1 - t_{01}) - \beta(t_{02} - T_2) \tag{6}$$

Defining the following dimensionless quantities:

$$R = \frac{\beta}{\alpha}, \quad S = \frac{\gamma}{\alpha}, \quad \tau = \frac{T_2}{T_1} \quad \text{and} \quad t_1 = \frac{t_{01}}{T_1} \tag{7}$$

applying the reversibility condition by inserting Q_1 and Q_2 into equation (4), and using the Carnot efficiency (equation (5)), the following equations for t_1 and W are obtained:

$$t_1 = \frac{1 - \eta + R\tau}{(1+R)(1-\eta) - S\eta^2} \tag{8}$$

$$\frac{W}{\alpha T_1} = 1 + R\tau - \frac{[(1-\eta) + R\tau][1 + R(1-\eta)]}{(1+R)(1-\eta) - S\eta^2} \tag{9}$$

The efficiency at maximum power is found by differentiating the power equation (equation (9)) with respect to η and setting the result to zero. This results in the following equation for the efficiency at maximum power (η_m):

$$\left[R^2(1+S\tau) + R(1+2S) + S\right]\eta_m^2 - 2\left[R^2(1+S\tau) + R(1+S+S\tau) + S\right]\eta_m + R(1+R)(1-\tau) = 0 \tag{10}$$

The above equation for η_m has two real roots with the one physically correct (i.e. nonnegative) given by

$$\eta_m = 1 - \frac{(1-\tau)RS}{R^2(1+\tau S) + R(1+2S) + S} - \sqrt{\left[1 - \frac{(1-\tau)RS}{R^2(1+\tau S) + R(1+2S) + S}\right]^2 - \frac{R(1+R)(1-\tau)}{R^2(1+\tau S) + R(1+2S) + S}} \tag{11}$$

It is interesting to note that when there is no leakage ($S = 0$), equation (11) above, irrespective of the value of R, reduces to

$$\eta_m = 1 - \sqrt{\tau} \tag{12}$$

which is the Curzon–Ahlborn efficiency [2]. The maximum work under these conditions is obtained from the following simple equation:

$$\frac{W_m}{\alpha T_1} = \frac{R}{R+1}\eta_m^2 \tag{13}$$

Therefore equation (9) (with W replaced by W_m and η by η_m) and (11) are the generalized forms of equations (13) and (12), respectively, reported by Curzon and Ahlborn [2]. In addition, equation (11), for $\tau = 0$ (i.e. $T_2 = 0$ since the value of T_1 is finite due to a negligible radiation) and with $R = S$ reduces to

$$\eta_m = 1 - \frac{R + \sqrt{(R+1)(R+2)}}{3R+2} \tag{14}$$

With $R = S = 1$, the efficiency at maximum power (equation (11)) can be written as a function of τ as follows:

$$\eta_m = 1 - \frac{1-\tau}{5+\tau} - \sqrt{1 - \frac{4(1-\tau)}{5+\tau} + \left(\frac{1-\tau}{5+\tau}\right)^2} \tag{15}$$

Finally, when R approaches infinity (i.e. when the only thermal resistance is between the working fluid and the high-temperature source), the value of η_m is governed by

$$\lim_{R\to\infty} \eta_m = 1 - \sqrt{\frac{\tau(1+S)}{1+\tau S}} \tag{16}$$

RESULTS AND DISCUSSION

The equations derived in the previous section are analysed here and the effects of the various parameters involved on the efficiency are discussed.

The conversion efficiency at maximum power (η_m) given by equation (11) is plotted in Figs 2–4 for $S = 0.1$, 0.5, and 1 as a function of R for values of τ ranging from 0 to 0.9. The general trend of the results is similar and shows η_m to increase with R for a given τ and to decrease with τ for a given R. This is to be expected since, when S and τ are both constant, increasing R (the ratio of heat transfer coefficients) reflects an increase in the heat transfer coefficient (or a decrease in resistance to heat flow) between the working fluid and the heat sink and results in a lower temperature for the cold reservoir. This, in turn, causes the Carnot-like engine to operate between a hot and cold reservoir of higher temperature difference and consequently, results in an increase of its efficiency. Furthermore, at constant values of R and S, a higher value of τ ($\tau = T_2/T_1$) means closer hot and cold reservoir temperatures (i.e. less efficient Carnot cycle) and hence a less efficient engine.

The effect of S on the efficiency at maximum power can be assessed by comparing results in Figs 2 to 4. As can be seen, increasing S decreases the value of η_m that can be achieved at specific values of R and τ due to higher losses from the engine. Furthermore, as R increases, η_m approaches a limiting value which may be evaluated from equation (16). In addition, equation (16) shows that for $\tau = 0$ this limiting value is independent of S and is always equal to 1. Therefore, all $\tau = 0$ curves displayed in Figs 2 to 4 approach 1 as R approaches infinity. Moreover, the variation of maximum work with R, S, and τ is found to be very similar to that of η_m and is presented, for completeness, in Fig. 5 for the case when $S = 1$. In Fig. 6, the

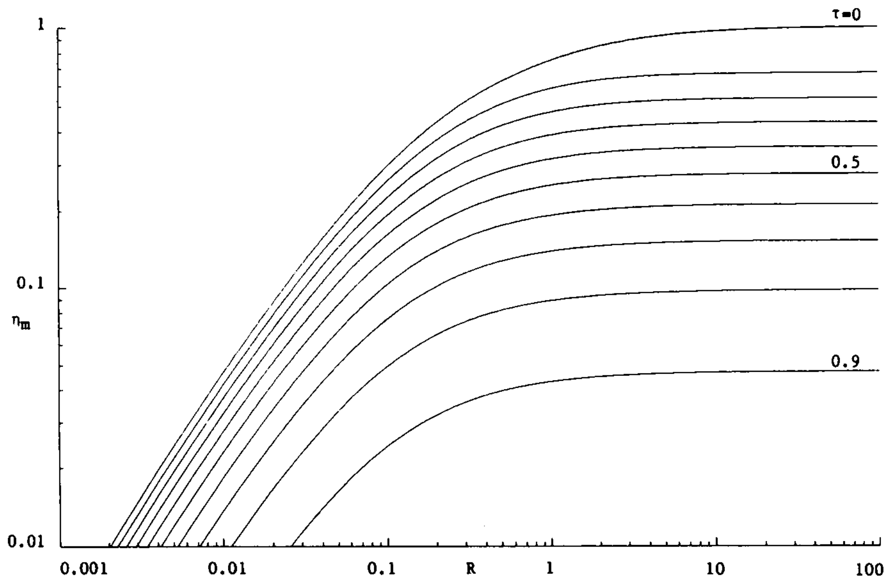

Fig. 2. Variation of the efficiency at maximum power with R for different values of $\tau (S = 0.1)$.

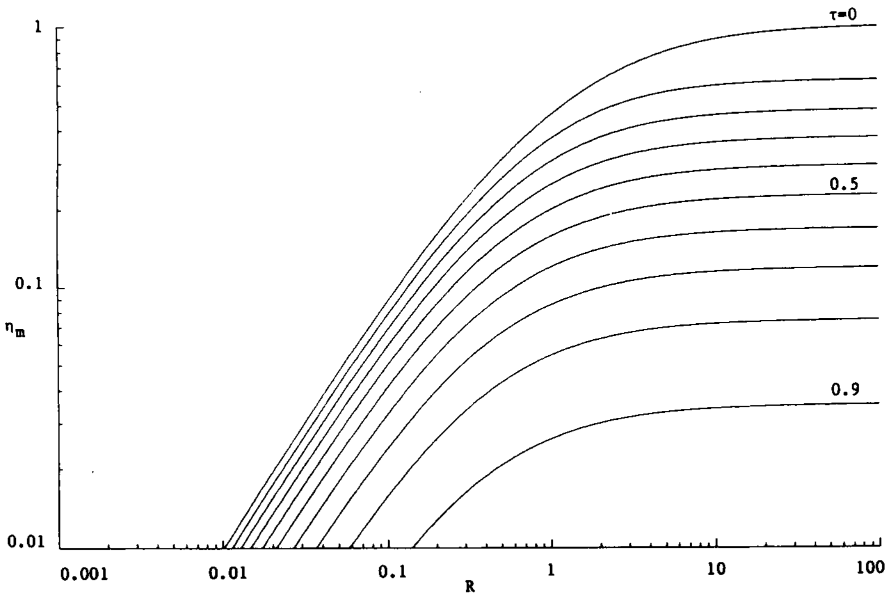

Fig. 3. Variation of the efficiency at maximum power with R for different values of $\tau (S = 0.5)$.

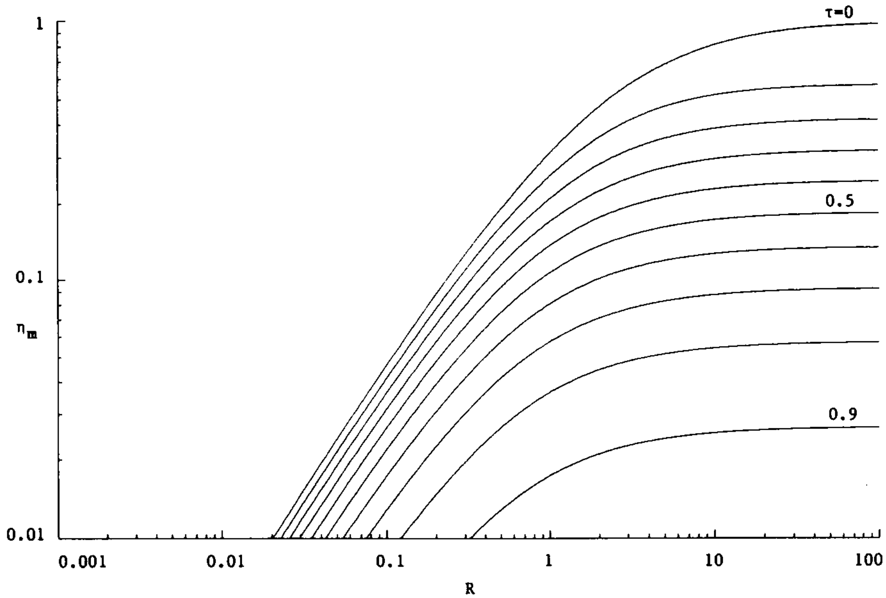

Fig. 4. Variation of the efficiency at maximum power with R for different values of $\tau\,(S = 1)$.

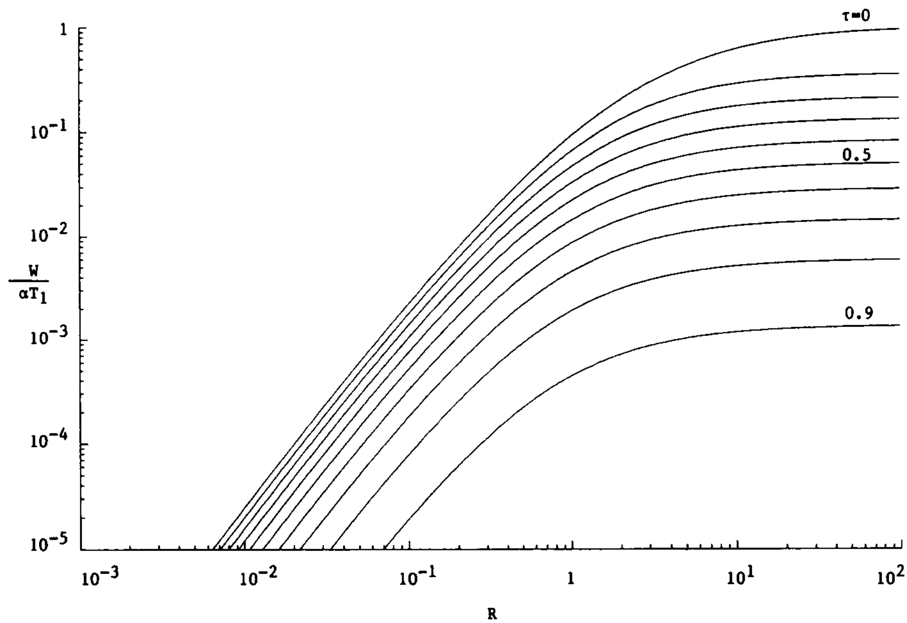

Fig. 5. Variation of maximum power with R for different values of $\tau\,(S = 1)$.

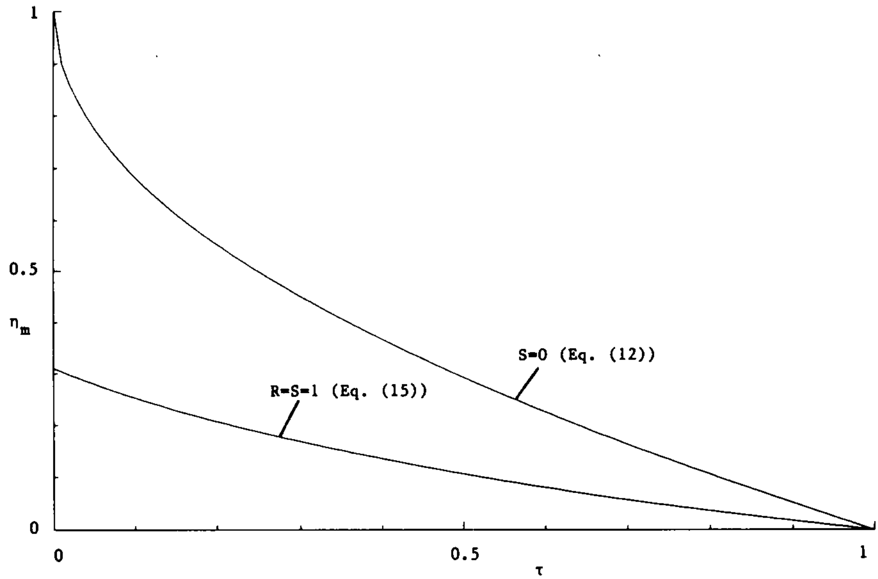

Fig. 6. Variation of the efficiency at maximum power with τ for $S = 0$ and $R = S = 1$.

Curzon–Ahlborn relation [2] and the efficiency at maximum power for $R = S = 1$ (the case of similar heat transfer coefficients for all heat transfer processes) are plotted as a function of τ. The realistic effect of heat leak on the efficiency is clearly seen by the decrease in the highest achievable efficiency with heat leak. Moreover, the variation of η_m with R for $\tau = 0$ when $R = S$ is depicted in Fig. 7. Again, the highest achievable efficiency is shown in this case to be more realistic and to approach a limiting value of 0.33 when R approaches infinity. In conclusion, leakage governs the highest achievable efficiency at maximum power.

CONCLUSION

An analytical investigation of the performance of Carnot-like engines with heat leak was undertaken. Heat transfer was assumed to occur via a conduction/convection mode and Newton's law of cooling was used to represent the heat transfer processes. Moreover, the Curzon–Ahlborn relation was generalized, the efficiency at maximum power was found to be limited by the heat-leak mechanism, and the predicted values were found to be more realistic. The simplicity and practical achievement of the model make it an attractive tool to students, who may use it for predicting the performance of a variety of energy conversion systems (e.g. power plants [10], thermoelectric generators [4], thermionic generators and so on).

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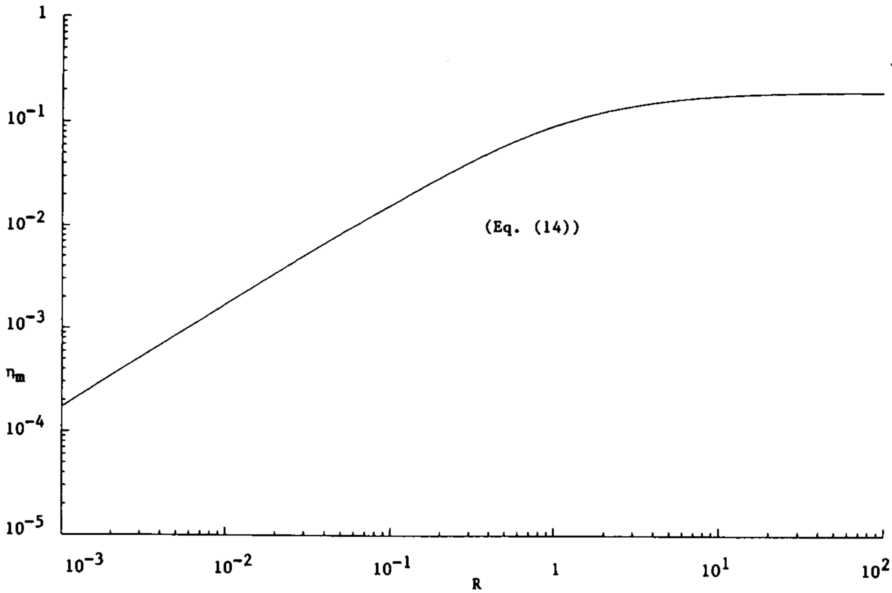

Fig. 7. Variation of the efficiency at maximum power with R for $R = S$ and $\tau = 0$.

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