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Evolution of power and entropy in a temperature gap system with electric and thermoelectric influences

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Abstract

Simple thermodynamic modeling of heat engines as thermal gap systems provides useful insight on the performance of power plants. When a temperature difference exists, a potential for power production ensues, and optimization is naturally sought. When electric and/or thermoelectric effects are present, the entropic behavior of such systems changes somewhat. An optimum power extraction temperature is found to be related to the so-called CNCA optimum temperature. The opportunity to generate thermoelectric power in a temperature gap is discussed in some detail in a fashion that renders it analogous to many current thermodynamic optimization studies.

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1. Introduction

For a given set of parametric constraints, the performance of a given system may be improved (power maximized in a power plant . . . etc.) by suitably setting certain parameters. This has been practiced in the disciplines known as finite time thermodynamics (FTT) [1] and entropy generation minimization (EGM) [2] or, more generally, exergy analysis (EA) [3]. There appears to be some disagreement between adherents to the above first two procedures [4–7]. Nevertheless, the aims of the two schools are similar and may best be described as thermodynamic optimization in search of enhanced performance. The disagreement is attributed to two basic aspects. The first is related to the nature of the interconnection between the “reversible internal” and the “irreversible

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external” components of the plant [4]. The other disagreement is on whether the rate of heat input may vary freely during internal optimization of the plant [6]. These issues shall not be discussed in this paper, and in any event, they are not of major concern since the systems studied here are inherently irreversible.

A temperature gap system is basically an “insulation” that separates two bodies (reservoirs), one of high temperature and the other of low temperature. Bejan [2] explained that a power plant (or a refrigeration plant) that receives heat from the hot body and expels heat to the cold body effectively functions as a thermal insulation. While the two thermal reservoirs do indeed exchange heat, they do not make thermal contact, since the plant separates them. The fact that they are in communication, although indirectly, presents an opportunity to generate power. In a regular gap (insulation), the heat flux is constant as the temperature drops from the high reservoir value to the low reservoir value. When a power plant is considered to be sandwiched between the reservoirs, the heat fluxes into and out of the plant are no longer conserved due to the opportunity for extraction of power.

The presence of an additional electric field influences the performance of a power producing device that can be modeled as a gap (insulation) between two temperature limits. An interesting occurrence is when the electric field owes its existence to the thermoelectric effect. The effect is usually neglected, since it is rather small compared to other effects, such as heat conduction. On the other hand, with semiconductors, the thermoelectric effects may be significant. Thus, a thermoelectric generator is unique in that the temperature gap and the “sandwiched” power plant are one and the same.

Assuming that classical concepts arising from EA, such as the Guoy–Stodola theorem, apply to such inherently irreversible systems, does provide some insight into the macroscopic thermodynamic relations involving thermoelectric power extraction. The driving motivation for this work is the attempt to ascertain the reasons for the relatively low efficiency of thermoelectric generators. In this paper, the generation of entropy in a temperature gap is first considered. The effect of the power extraction temperature and its location are then discussed. The inclusion of electric and thermoelectric effects in the temperature gap system is the final consideration. Some conclusions pertinent to thermoelectric power generation are presented with some questions remaining open for further research.

2. Generation of entropy and power production

A temperature gap system (also called an insulation) is considered to be sandwiched between a heat source at T_H and a heat sink at T_L . The heat flow $\dot{Q}$ proceeds undiminished across the system length. Recalling the second law for a closed system (the gap), the entropy generation rate is

$$\dot{S}_{\text{gen}} = \dot{Q} \left(-\frac{1}{T_H} + \frac{1}{T_L} \right) = K \left[\frac{(T_H - T_L)^2}{T_H T_L} \right] = \frac{kA}{L} \left[\frac{(T_H - T_L)^2}{T_H T_L} \right] \quad (1)$$

where it has been assumed that the heat proceeds by an assumed linear conductive process, and the “insulation” has a conductance K . In the last expression, it is assumed that the conductance is that of a material with a thermal conductivity k , area A and length L .

Applying a steady exergy balance leads to the Guoy–Stodola theorem on destruction of availability [3]. The power produced is, thus, equal to the maximum available energy less the destroyed availability (the destroyed exergy). This can be stated (referenced to the immediate surroundings) as

$$\dot{W} = \dot{Q}\eta_c - T_L \dot{S}_{\text{gen}} \quad (2)$$

When $\dot{S}_{\text{gen}}$, the rate of entropy generation between T_H and T_L , is inserted (i.e. using Eq. (1)), the power produced is zero, as is expected for a pure heat transport situation.

Thus, given a temperature gap ($T_H - T_L$), the maximum entropy generation rate is equivalent to losing all available work potential (exergy). By reducing the entropy generation rate, some work may be recovered. The theoretical upper bound on the recoverable power is when $\dot{S}_{\text{gen}}$ is zero, and the Carnot limit is approached.

3. Optimal insulation/gap cooling location in a temperature gap

Following Bejan [2], an insulation system of length L , area A and thermal conductivity k operating between a heat source at T_H and heat sink at T_L is considered. At some intermediate temperature T_m and location x , heat is removed by the inclusion of a reversible engine that operates between that temperature and the sink temperature. Fig. 1a shows the gap, and Fig. 1b shows the reversible engine extracting heat from the gap at temperature T_m .

The entropy generation is “internal” to the gap system, and its rate is given by

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}_L}{T_L} + \frac{\dot{Q}_H - \dot{Q}_L}{T_m} - \frac{\dot{Q}_H}{T_H} \quad (3)$$

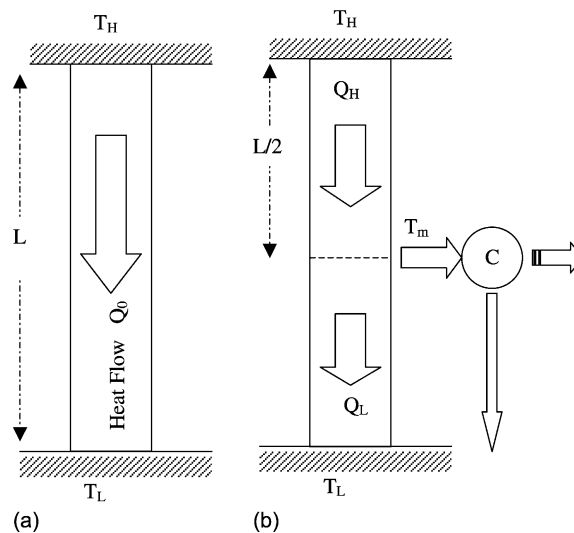


Fig. 1. (a) Schematic of temperature gap system with no power extraction and (b) temperature gap system with power extraction at some intermediate temperature.

Consider that

$$\dot{Q}_H = \frac{kA}{x}(T_H - T_m) \quad \text{and} \quad \dot{Q}_L = \frac{kA}{L-x}(T_m - T_L) \quad (4)$$

are the heat flows towards and away from the extraction point, respectively.

Combining these relations, $\dot{S}_{\text{gen}}$ emerges as a function of T_m and x . Minimizing this function by taking $\partial\dot{S}_{\text{gen}}/\partial T_m$ and $\partial\dot{S}_{\text{gen}}/\partial x$ results in:

$$x_{\text{opt}} = \frac{L(T_H - T_m)\sqrt{T_L}}{(\sqrt{T_H} - \sqrt{T_L})(\sqrt{T_H T_L} + T_m)} \quad \text{and} \quad T_{m,\text{opt}} = \sqrt{\frac{x(T_L - T_H)T_L T_H + L T_L T_H^2}{x(T_H - T_L) + L T_L}} \quad (5)$$

If the above two equations are solved simultaneously for x , the result is $x_{\text{opt}} = L/2$, which indicates that, indeed, midpoint “cooling” produces minimum entropy generation. If from the onset (as in Bejan [2]), the geometric midpoint were selected, the result is found by putting $x = L/2$ in the second of Eq. (5), producing:

$$T_{m,\text{opt}} = \sqrt{T_H T_L} \quad (6)$$

Eq. (5) are, however, more general in that they give the optimal heat extraction temperature at any location along the system and additionally give the optimal position for any length and any selected heat extraction location (T_m). That the above is analogous to maximum power needs further consideration.

4. Power extraction from a temperature gap with external heat transfer irreversibilities

When a certain amount of power is extracted from a temperature gap ($T_H - T_L$), the maximum heat input is no longer what was $\dot{Q}$ in Section 1. The available heat input is now $\dot{Q}_H$, which depends on the power extraction temperatures within the gap.

The entropy generation in the gap of Fig. 1b is given by summing the entropy generation from T_H to T_1 and from T_2 to T_L with no other contribution, assuming internal reversibility, and where T_1 and T_2 are the hot and cold side temperatures of the power extraction compartment. Given that $\dot{Q}_H = K(T_H - T_1)$ is the heat entering the gap and $\dot{Q}_L = K(T_2 - T_L)$ is the heat exiting it, the power produced according to the Guoy–Stodola theorem (steady exergy balance with T_L as reference) is

$$\dot{W} = \dot{Q}_H \eta_c - T_L \left\{ \dot{Q}_H \left(-\frac{1}{T_H} + \frac{1}{T_1} \right) + \dot{Q}_L \left(-\frac{1}{T_2} + \frac{1}{T_L} \right) \right\} \quad (7)$$

Inserting the relations for $\dot{Q}_H$ and $\dot{Q}_L$, this can be optimized with respect to T_1 and T_2 to give $T_{1,\text{opt}} = \sqrt{T_H T_L}$ and $T_{2,\text{opt}} = T_L$.

The maximum obtainable power is found by inserting the above optimum temperatures into

$$\dot{W}_{\text{max}} = \dot{Q}_H \left(1 - \sqrt{\frac{T_L}{T_H}} \right) \quad (8)$$

or in terms of conductance and temperature only:

$$\dot{W}_{\text{max}} = K T_H \left(1 - \sqrt{\frac{T_L}{T_H}} \right)^2 \quad (9)$$

which is the well known so-called endoreversible CNCA result [8].

It is noted that if Eq. (7) is expanded in terms of $\dot{Q}_H$ and $\dot{Q}_L$, we obtain,

$$\dot{W}_{\max} = \dot{Q}_H \left(1 - \frac{T_L}{T_1}\right) - \dot{Q}_L \left(1 - \frac{T_L}{T_2}\right) \quad (10)$$

Eq. (10) can be obtained by applying a steady exergy balance on the aggregate system sandwiched between T_H and T_L .

If now, the optimum values of T_1 and T_2 are inserted, what emerges is that the maximum power is equivalent to a model that has a reversible engine operating between the optimum value of T_1 and T_L reversibly when supplied with a heat input of $\dot{Q}_H$. The maximum power is

$$\dot{W}_{\max} = \dot{Q}_H \left(1 - \frac{T_L}{T_{1,\text{opt}}}\right) \quad (11)$$

This result can be interpreted as the observation that the heat flow $\dot{Q}_H$ drives the maximum availability (i.e. in the absence of entropy generation, $\dot{Q}_H$ flows undiminished from $T_{1,\text{opt}}$ to T_L rather than from T_1 to T_2).

5. Power extraction evolution from a temperature gap with specified extraction temperature

A model that involves internal irreversibility is now sought. Using the model of the previous section, T_1 may be assumed equal to T_2 and equal to some midpoint temperature T_m . Inserting this into Eq. (7) gives the power produced by a plant (viewed as a temperature gap) that is externally reversible but internally irreversible. Now, if the resulting equation for power is differentiated with respect to T_m , the result would indicate that the optimum temperature is $T_{m,\text{opt}} = \sqrt{T_L \bar{T}}$ (where $\bar{T}$ is the average temperature in the gap). This result is equivalent to a model where an engine is extracting an amount of power from a heat flow of $\dot{Q}_H - \dot{Q}_L$ rather than $\dot{Q}_H$ and is operating reversibly between T_m and T_L . As a result, it is seen that this model is neglecting a significant part of the available energy (exergy) in the gap.

The previous model can be improved by considering that the maximum available heat is not $\dot{Q}_H$ (which depends on T_m) but rather $\dot{Q}$ (the undiminished heat flow from T_H to T_L in the absence of any power extraction). The power is once again given by the steady state Guoy–Stodola theorem according to Eq. (7), but replacing $\dot{Q}_H$ with $\dot{Q}$. Considering that $\dot{Q}$ is not dependent now on T_1 or T_2 , the optimum is when T_1 is equal to T_H and T_2 is equal to T_L . This, obviously, produces the Carnot limit, which is not being questioned here.

Now, considering that $T_1 = T_2 = T_m$ in a midpoint engine model, we obtain:

$$\dot{W} = \dot{Q}\eta_c - T_L \left\{ \dot{Q}_H \left(-\frac{1}{T_H} + \frac{1}{T_m} \right) + \dot{Q}_L \left(-\frac{1}{T_m} + \frac{1}{T_L} \right) \right\} \quad (12)$$

Re-writing in a dimensionless form:

$$\dot{W}^* = \frac{\dot{W}}{\frac{kA}{L} T_H} = (1 + \tau - 2\tau_m) \left(1 + \tau - 2\frac{\tau}{\tau_m} \right) \quad (13)$$

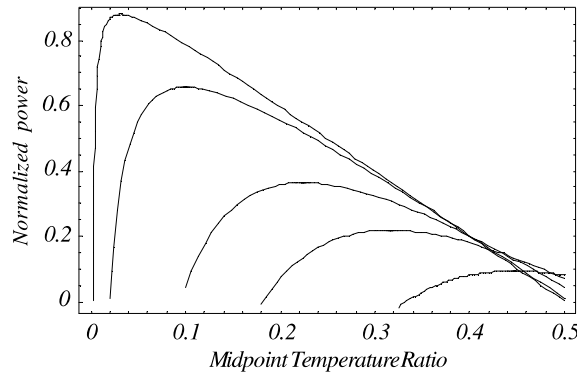


Fig. 2. Power evolution in gap system as a function of the midpoint temperature ratio for a given overall temperature ratio.

where τ_m is the ratio of the midpoint temperature to the heat source temperature (T_m/T_H), and τ is the ratio of the low temperature to the heat source temperature (T_L/T_H).

Fig. 2 shows a plot of the non-dimensional power versus τ_m for different temperature ratios (T_L/T_H). The figure immediately reveals that power can be extracted from a gap (T_H to T_L) when the midpoint temperature (in this model) is within a specified range. This range is from the arithmetic mean temperature (no power—simply heat conduction) to the harmonic mean temperature (entropy generation at a maximum). Within this narrow range, an optimum exists at $\sqrt{T_L T_H}$. Beyond this range, to extract power, either cooling (left) or heating must be occurring. The curves shift to the right and become narrower as the temperature ratio increases.

In fact, the setup presented may be viewed as equivalent to a model of power plants proposed by Bejan in Ref. [2]. The simplest form is a given heat input ($\dot{Q}$) supplying two parallel paths: a “bypass heat leak” part which is not passive but produces some power, and, in parallel, a “Carnot” compartment. The heat ($\dot{Q}_H$) is routed to the heat leak, while ($\dot{Q} - \dot{Q}_H$) enters the Carnot compartment. Clearly then, the total power must be the sum of the contribution of the two parallel parts of the system:

$$\dot{W} = (\dot{Q} - \dot{Q}_H)\eta_c - (\dot{Q}_H - \dot{Q}_L)\left(1 - \frac{T_L}{T_m}\right) \quad (14)$$

where $\dot{Q}_L$ is the heat rejected from the bypass heat leak and T_m is the temperature at which the bypass part extracts power. It will be shown soon that this does indeed produce the same optimum midpoint temperature as above. Eq. (14) can be expanded and found to reproduce Eq. (12). This verifies that the models are consistent and also conform to the Guoy–Stodola theory on exergy destruction. Thus, it could be stated that “for an internally irreversible plant (due to the heat transfer, which is inherent), the useful power that may be produced is equal to the total unrecoverable availability of the thermal gap minus the internally generated exergy destruction”.

What is being discussed here is that the bypass heat leak from T_H to T_L is itself “leaky” and feeds to the Carnot engine. Bejan [2] modeled a more complex system that has a continuously “leaky heat leak,” which produces power between every local leak from the bypass heat leak and the heat sink. Nuwayhid et al. [9] considered a simplified model that considers the

bypass heat leak to “leak” heat to the Carnot engine at only one location (the midpoint in this case). This model was found to give somewhat higher power, but was certainly bound by the Carnot limit. It will be used later, although some caution has to be practiced when using the results.

6. Entropy generation in an insulation in the presence of an electric current

In addition to heat conduction in the simple thermal gap system, Joule heating is now included in the analysis to study what are the possible effects ensuing. As a starting point, it is instructive to assume that the total Joule heat in the “insulation” is apportioned equally above and below the geometric midpoint, which is at a temperature T_m . While the entropy generation rate ($\dot{S}_{\text{gen}}$) is still given by Eq. (3), the heat transfers must include the added Joule heating effect. Assuming equipartition of the total Joule heat, the heat input and output rates become:

$$\dot{Q}_H = \frac{kA}{L/2} (T_H - T_m) - \frac{1}{2} I^2 \frac{\rho L}{A} \quad \text{and} \quad \dot{Q}_L = \frac{kA}{L/2} (T_m - T_L) + \frac{1}{2} I^2 \frac{\rho L}{A} \quad (15)$$

where ρ is the electrical resistivity of the gap material.

The entropy generation rate is, thus, given by

$$\dot{S}_{\text{gen}} = 2K \left(\frac{T_m}{T_L} + \frac{T_L}{T_m} + \frac{T_H}{T_m} + \frac{T_m}{T_H} - 4 \right) + \frac{RI^2}{2} \left(\frac{1}{T_L} + \frac{1}{T_H} - \frac{2}{T_m} \right)$$

where $R = \rho L/A$ is the total electrical resistance, and $K = kA/L$ is the total thermal conductance of the insulation.

Differentiating with respect to T_m leads to

$$T_{m,\text{opt}} = \sqrt{T_H T_L \left(1 - I^2 \frac{\rho L^2}{2kA^2} \frac{1}{(T_H + T_L)} \right)} = \sqrt{T_H T_L - \frac{1}{4} \frac{T_H T_L (T_H - T_L)}{(T_H + T_L)} \frac{I^2 R}{K(T_H - T_L)}} \quad (16)$$

It is observed that in the absence of an electric field, the usual relation is found. It is also seen that the optimum temperature of power extraction is reduced as more current flows and this can be analyzed as a function of the ratio of Joule-to-conduction heat.

The above equation can also be written in dimensionless form:

$$\tau_{m,\text{opt}} = \sqrt{\tau \left(1 - \frac{\beta}{2(1 + \tau)} \right)} \quad (17)$$

where $\tau_m = T_m/T_H$; $\tau = T_L/T_H$; $\beta = \rho I^2 L^2 / kA^2 T_H$.

Since τ_m is a fraction such that $\tau < \tau_m < 1$, hence $\beta \leq 2(1 - \tau^2)$, and the maximum allowed current parameter is thus:

$$\beta_{\text{max}} = 2(1 - \tau^2) \quad (18)$$

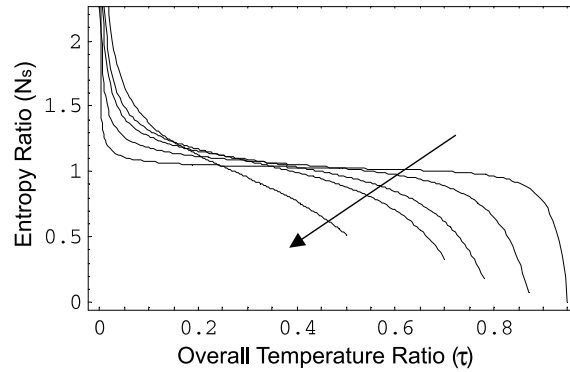


Fig. 3. Entropy generation number versus overall temperature ratio for different electric current parameter.

Hence, the minimum entropy generation rate is

$$\dot{S}_{\text{gen opt}} = -8K + \left(\frac{T_H + T_L}{T_L T_H} \right) + \left(\frac{I^2 R}{2} + 2K \sqrt{T_H T_L \left(4 - \frac{2I^2 R}{K(T_H + T_L)} \right)} \right) \quad (19)$$

or in the dimensionless form:

$$\frac{\dot{S}_{\text{gen opt}}}{K} = \frac{\beta - 16\tau + \beta\tau + 4(1 + \tau) \sqrt{\tau \left(4 - \frac{2\beta}{1 + \tau} \right)}}{2\tau} \quad (20)$$

An entropy generation number N_S may be defined as the ratio of entropy generation with an electric current present to that when no electric current exists.

$$N_S = \frac{S_{\text{gen opt}}}{S_{\text{gen opt...}I=0}} = \frac{\beta - 16\tau + \beta\tau + 4(1 + \tau) \sqrt{\tau \left(4 - \frac{2\beta}{1 + \tau} \right)}}{-16\tau + 8\sqrt{\tau}(1 + \tau)} \quad (21)$$

Fig. 3 shows N_S versus τ for several values of β within the acceptable range $0 < \beta < 2(1 - \tau^2)$ (which assures that τ_m remains in the appropriate range defined above). When no electric current is present $N_S = 1$. There exists a temperature above which the entropy generation will be less than for the case with no electric current (for a given value of β). In general, as β increases, the entropy generation decreases within the appropriate range for τ (which also diminishes).

Considering an endoreversible power generation model with finite rate heat transfer from a heat source, the hot side of the device (including electric current flow) may be selected as given by Eq. (16) for maximum output. The condition reverts to the usual CNCA [8] condition in the absence of the electric field ($I = 0$) and shows a somewhat lower optimum temperature.

7. Inclusion of thermoelectric effects

In the previous section, the effect of an electric current was shown. The existence of an electric current can be attributed to the presence of a temperature gradient. This brings to light the

thermoelectric effect. To start with, let us simplify the situation by assuming a single material “the insulation” with the usual heat conduction occurring but with the added allowance for the thermoelectric effect. The entropy generation rate in this insulation (plant, thermoleg...etc.) is still given by Eq. (1), but the local heat flow is now as follows [10]:

$$\dot{Q}(x) = \alpha IT(x) - kA \frac{dT}{dx} \quad (22)$$

where one dimensional geometry has been assumed (and is more or less justified), the Thomson effect is neglected (by using average Seebeck values) and α is the Seebeck coefficient. The equation assumes the hot side (T_H) is at location $x = 0$ and the sink side (T_L) is at $x = L$. The equation shows the ordinary conductive mechanism in addition to the Peltier transport of heat by the charge system (electrons or holes depending on the type of material). The Peltier heat flow is seen to always oppose the conduction flow.

It is clear that the temperature profile in the insulation is required. The presence of Joule heating due to the electric currents' passage makes the profile non-linear. The steady state energy equation is solved with properties taken constant and the Thomson heat neglected:

$$kA \frac{d^2 T}{dx^2} = -\frac{\rho I^2}{A} \quad (23)$$

For the problem involved, the solution is

$$T(x) = T_H - \frac{x}{L}(T_H - T_L) - \frac{\rho I^2}{2kA^2}x(x - L) \quad (24)$$

The gradient of the temperature is

$$\frac{dT}{dx} = -\frac{(T_H - T_L)}{L} - \frac{\rho I^2}{2kA^2}(2x - L) \quad (25)$$

Substituting into the equation for $\dot{Q}(x)$ gives the local heat flow:

$$\dot{Q}(x) = \alpha IT(x) + kA \frac{(T_H - T_L)}{L} + \frac{1}{2} \rho \frac{I^2}{A}(2x - L) \quad (26)$$

In the absence of thermoelectric effects, the heat flow proceeds unimpeded from T_H to T_L , and the entropy generation rate is given by Eq. (1). On the other hand, if thermoelectric effects are included, Eq. (1) is no longer valid since $\dot{Q}(x)$ is location dependent as given by Eq. (26).

The entropy generation rate (in one dimension) is given by [2]:

$$\dot{S}_{\text{gen}} = \frac{1}{T} \frac{d\dot{Q}}{dx} - \frac{1}{T^2} \dot{Q} \frac{dT}{dx} \quad (27)$$

Using Eqs. (24)–(26) in Eq. (27), the local entropy generation rate (W/m K) is now:

$$\dot{S}_{\text{gen}}(x) = kA \frac{4k^2 A^4 + 4I^2 k A^2 \rho [LT_H + (x - L)\Delta T] + I^4 \rho^2 L^2 (L^2 + 2x^2 - 2Lx)}{(2kA^2 [LT_H - x\Delta T] + I^2 \rho Lx(L - x))^2} \quad (28)$$

No attempt has been made to simplify this any further. Fig. 4 shows the entropy generation rate along the leg length for different temperature differences. The maximum entropy generation is clearly on the cold side, while the curvature slowly increases as the driving temperature difference

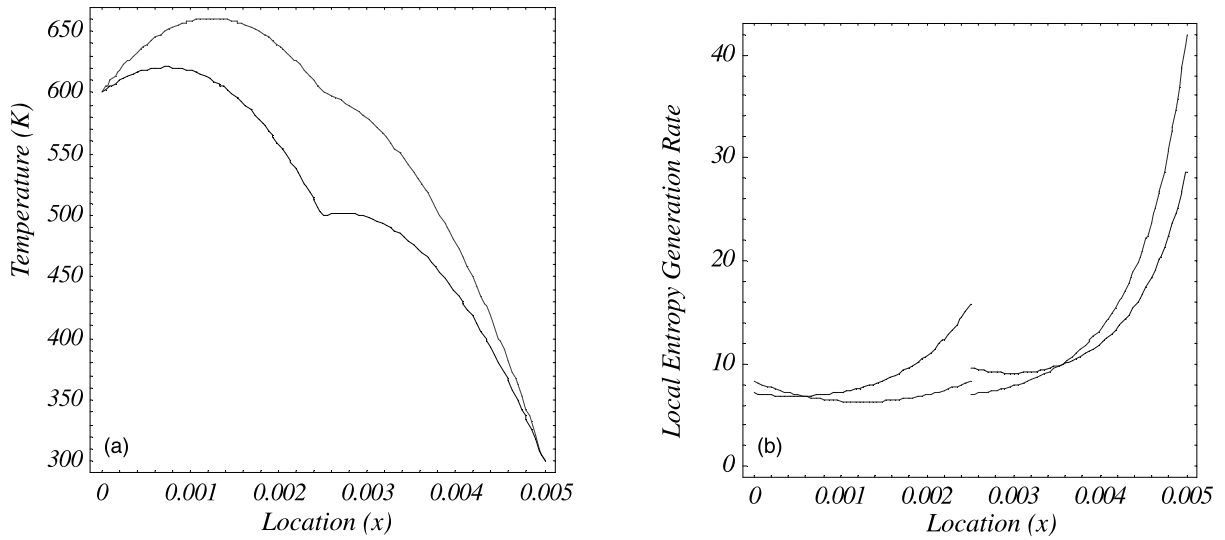


Fig. 4. (a) Temperature profile along thermoleg. (b) Local entropy generation rate along thermoleg (for given geometry, material properties and electric current).

increases. The trivial conclusion that decreasing the heat sink temperature reduces entropy generation is apparent.

The total entropy generation in the “insulation” (e.g. a single thermoleg) is found by integrating Eq. (28) from $x = 0$ to $x = L$. This yields:

$$\dot{S}_{\text{gen,leg}} = \frac{kA}{L} \frac{(T_H - T_L)^2}{T_H T_L} + \frac{\rho I^2}{2} \frac{L}{A} \frac{(T_H + T_L)}{T_H T_L} \quad (29)$$

This result can also be obtained by evaluating:

$$\dot{S}_{\text{gen,leg}} = -\frac{\dot{Q}_H}{T_H} + \frac{\dot{Q}_L}{T_L} \quad (30)$$

where $\dot{Q}_H$ and $\dot{Q}_L$ are heat flows evaluated from Eq. (26) at $x = 0$ and at $x = L$, respectively, and it is noted that the presence of thermoelectricity causes the heat flow rates to be location dependent (as given by Eq. (26)) and therefore, different in the upper and lower parts of the leg. Eq. (30) can be obtained by imposing a simple entropy balance on the leg [11]. Clearly from Eq. (29), in the absence of an electric field, the entropy generation rate is simply that for a “conductive” insulation system.

While it is noted that what has been discussed considers only internal irreversibilities, an entropy generation number (N_S) based on the internal entropy generation can be defined to be the entropy generation in the presence of thermoelectricity to the entropy generation due to pure conduction of heat:

$$N_S = 1 + I^2 \frac{\rho}{k} \left(\frac{L}{A} \right)^2 \frac{\bar{T}}{(T_H - T_L)^2} \quad (31)$$

where the average temperature is $\bar{T} = (T_H + T_L)/2$.

Eq. (31) shows that for a given material and a given electric current, entropy generation is least when the temperature difference is largest or if the length-to-area ratio is smallest. On the other hand, for a given geometry and temperature difference, entropy generation is small when the electric resistivity-to-thermal conductivity ratio is small. Conductors have a low ρ/k ratio, while insulators have a large ρ/k . Semiconductors are somewhere in between (e.g. Copper 4×10^{-9} , Bi_2Te_3 9×10^{-6} Alumina $3 \times 10^{11} \text{ m}^2 \text{ K/A}^2$).

Obviously, entropy generation is less when there is no electric current. On the other hand, since the electric current is being sought and is a result of the temperature gradient (Seebeck effect), a zero current case is a trivially useless result. Optimization is, therefore, sought in order to maximize either efficiency or electric power.

Considering a full two-legged thermocouple provides more insight. In such a situation, one can write for each leg (n and p type) an equation for the heat flow similar to Eq. (26) but with the heat flow at a location x being defined as that in *both* legs:

$$\dot{Q}(x) = 2\alpha IT(x) + 2kA \frac{(T_H - T_L)}{L} + \rho \frac{I^2}{A} (2x - L) \quad (32)$$

where it has been assumed that $|\alpha_p| \approx |\alpha_n|$, $k_p \approx k_n$ and $\rho_p \approx \rho_n$ and it is noted that α_p is positive while α_n is negative. Additionally, the equation assumes the same geometry in both legs and a similar temperature profile.

If Eq. (31) is evaluated both at $x = 0$ and at $x = L$ and the resulting Q_H and Q_L subtracted, what emerges is the anticipated result that the power is given by $\dot{W} = 2\alpha I(T_H - T_L) - 2(\rho L/A)I^2$, that is, the power is equal to the open circuit power less the total Joule heat losses. Optimization of power with respect to current would give the maximum power to be

$$\dot{W}_{\max} = \frac{1}{2} \frac{\alpha^2}{\rho} \frac{A}{L} (T_H - T_L)^2$$

In general, one could define a load ratio ($m = R_L/R$, where $R = \rho L/A$ and R_L is the external “load” resistance) so that maximum power would be given as a function of m as

$$\dot{W} = \frac{2m}{(1+m)^2} \frac{\alpha^2}{\rho} \frac{A}{L} (T_H - T_L)^2$$

When $m = 1$, $\dot{W}_{\max}$ is produced.

It should be noted that the optimum current relation is of the same form for a single material as for a couple, consecutively, since in $I = \alpha \Delta T / (\rho L/A)(1+m)$, the α and ρ refer to the couple in the second case (i.e. $\alpha_{\text{couple}} = 2\alpha$ and $\rho_{\text{couple}} = 2\rho$).

If entropy generation is considered by utilizing Eq. (30), the result is that the total entropy generation rate in the two legs is simply that of Eq. (29) multiplied by a factor of 2. This leads to the same entropy generation number for the whole couple as is given for a single leg by Eq. (30). This is to be expected, since the two legs are assumed to have similar properties and temperature profiles.

Returning to Eq. (31) and replacing I using the above information gives:

$$N_S = 1 + \frac{Z\bar{T}}{(1+m)^2} \quad (33)$$

This says that the additional fractional entropy generated is unavoidably due to the obtained electric output as described by the *dimensionless figure of merit* (ZT), where the figure of merit is $Z = \alpha^2 / \rho k$ for a single material. The next section expands on this.

8. Inclusion of heat dumping in entropy generation

The result of Eq. (29) does not produce the anticipated optimization with respect to current. To remedy this, it has to be realized that $\dot{Q}_H$ must be allowed to vary. The simplest equivalent model is to consider an inexhaustible heat source ($\dot{Q}^*$) at T_H , supplying the device with a heat $\dot{Q}_H$, while the remainder ($\dot{Q}^* - \dot{Q}_H$) is rejected (dumped). The entropy generation is now:

$$\dot{S}_{\text{gen}} = -\frac{\dot{Q}_H}{T_H} + \frac{\dot{Q}_L}{T_L} + (\dot{Q}^* - \dot{Q}_H) \left(-\frac{1}{T_H} + \frac{1}{T_L} \right) = \dot{Q}^* \left(\frac{\Delta T}{T_H T_L} \right) - \frac{\dot{Q}_H - \dot{Q}_L}{T_L} \quad (34)$$

Inserting the appropriate heat flow at T_H and T_L from Eq. (26) gives:

$$\dot{S}_{\text{gen}} = \dot{Q}^* \left(\frac{\Delta T}{T_H T_L} \right) - 2\alpha I \frac{\Delta T}{T_L} + I^2 \frac{2\rho L}{A} \frac{1}{T_L} \quad (35)$$

Replacing the current with the aid of the load ratio by its optimum value:

$$\dot{S}_{\text{gen,min}} = \dot{Q}^* \left(\frac{\Delta T}{T_H T_L} \right) - \frac{2m}{(1+m)^2} \frac{\alpha^2}{\rho} \frac{A}{L} \frac{\Delta T^2}{T_L} \quad (36)$$

Minimum entropy generation (and therefore maximum power) will be produced when $m = 1$. This can be verified by differentiation of Eq. (36) with respect to m . As a reminder it is noted that the same previous observations are also valid.

A form of entropy generation number can be defined by dividing Eq. (36) by the entropy generation in the case of the absence of thermoelectricity (but with heat dumping), which is $\dot{S}_{\text{gen,noTE}} = \dot{Q}^* (\Delta T / T_H T_L)$. This gives:

$$N_{S,TE} = 1 - \frac{\dot{Q}_H - \dot{Q}_L}{\dot{Q}^*} \frac{T_H}{\Delta T} = 1 - \frac{2m^2}{(1+m)^2} \frac{\alpha^2}{\rho} \frac{A}{L} \frac{T_H \Delta T}{\dot{Q}^*} = 1 - \frac{2m^2}{(1+m)^2} f Z T_H \quad (37)$$

where $f = (kA\Delta T/L)/\dot{Q}^*$ is a form of dimensionless heat leak fraction that indicates how much of the input energy is conducted through the device. For a given heat leak, the larger the dimensionless figure of merit (ZT_H), the less entropy is generated and the more power is produced. On the other hand, since both Z and f depend on k (inversely), it may be more appropriate in the case of an inexhaustible energy supply ($\dot{Q}^*$) to study the power factor (α^2/ρ). Thus, the greater the power factor, the less entropy (more power) is generated for a given temperature difference and source temperature. Obviously, semiconductors have the highest Z and therefore, will produce the most power.

It is worth noting that including heat dumping in the energy balance (PM method) does not affect the result. This can be shown as follows:

$$\dot{W} = \dot{Q}^* - \dot{Q}_L - \dot{Q}_e = \dot{Q}^* - \dot{Q}_L - (\dot{Q}^* - \dot{Q}_H) = \dot{Q}_H - \dot{Q}_L \quad (38)$$

9. Temperature profile manipulation in a thermoelectric leg

9.1. Entropy generation due to midpoint variation

A material gap of length L is considered to be located between a heat source at T_H and a heat sink at T_L . The material gap is considered to be cooled to a temperature T_m at the geometric midpoint ($L/2$) as suggested in Bejan [2] for a purely conductive gap (Fig. 1a). The inclusion of thermoelectric effects means that the heat flow and the temperature profile are not uniform along the length. The heat flow is described by Eq. (22). The temperature profiles above and below the cooling point must now be reevaluated. Integrating Eq. (23) first from $x = 0$ to $x = L/2$ and then from $x = L/2$ to $x = L$ gives:

$$\begin{aligned} T(x) &= T_H - \frac{x}{L/2}(T_H - T_m) - \frac{\rho I^2}{2kA^2}x(x - L/2) \quad 0 \leq x \leq L/2 \\ T(x) &= T_L + \frac{(L-x)}{L/2}(T_m - T_L) - \frac{\rho I^2}{2kA^2}(x - L)(x - L/2) \quad L/2 \leq x \leq L \end{aligned} \quad (39)$$

The local (position dependent) heat fluxes in the two segments are now given by

$$\begin{aligned} \dot{Q}_H(x) &= \alpha IT(x) + kA \frac{(T_H - T_m)}{L/2} + \frac{1}{2}\rho \frac{I^2}{A} \left(2x - \frac{L}{2}\right) \quad 0 \leq x \leq L/2 \\ \dot{Q}_L(x) &= \alpha IT(x) + kA \frac{(T_m - T_L)}{L/2} + \frac{1}{2}\rho \frac{I^2}{A} \left(2x - \frac{3L}{2}\right) \quad L/2 \leq x \leq L \end{aligned} \quad (40)$$

A check on the above can be provided by requiring that $\dot{Q}_H(x)|_{x=0} = \dot{Q}_0(x)|_{x=0}$ where $\dot{Q}_0$ is the heat flux given by Eq. (26). The result would indicate that the equality holds when the midpoint temperature is

$$T_m = \bar{T} + \frac{1}{8}I^2 \frac{\rho}{k} \left(\frac{L}{A}\right)^2 = \bar{T} + \frac{1}{8}Z \frac{\Delta T^2}{(1+m)^2} \quad (41)$$

where the current has been replaced by invoking Ohms law and introducing the thermoelectric figure of merit (Z) and the load ratio (m). Eq. (41) shows that the midpoint temperature is normally higher than the average (linear) temperature, depending on the current and other parameters in the second term on the right. This same equation can be obtained from Eq. (24) with $x = L/2$.

Now, knowing T , dT/dX , $Q(x)$ and dQ/dT , the local entropy generation rate is found from Eq. (27). The following two equations give the entropy generation rate as a function of position along the leg:

$$\dot{S}_{\text{gen,hot}}(x) = kA \frac{64k^2A^4\Delta T_{Hm}^2 + 16I^2\rho kA^2L^2(T_m + 2\frac{x}{L}\Delta T_{Hm}) + I^4\rho^2L^4\left(1 - 4\frac{x}{L} + 8\left(\frac{x}{L}\right)^2\right)}{\left(4kA^2L(T_H - 2\frac{x}{L}\Delta T_{Hm}) + I^2\rho L^2\frac{x}{L}(1 - 2\frac{x}{L})\right)^2} \quad (42)$$

$$\dot{S}_{\text{gen,cold}}(x) = kA \frac{16I^2\rho kA^2L^2\left((2\frac{x}{L} - 1)\Delta T_{mL} - T_L\right) - 64k^2A^4\Delta T_{mL}^2 + I^4\rho^2L^4\left(5 - 12\frac{x}{L} + 8\left(\frac{x}{L}\right)^2\right)}{\left(4kA^2L((2\frac{x}{L} - 1)\Delta T_{mL} + \Delta T_{Hm} - T_H) + I^2\rho L^3\left(1 - 3\frac{x}{L} + 2\left(\frac{x}{L}\right)^2\right)\right)^2} \quad (43)$$

where $\Delta T_{Hm} = T_H - T_m$ and $\Delta T_{mL} = T_m - T_L$.

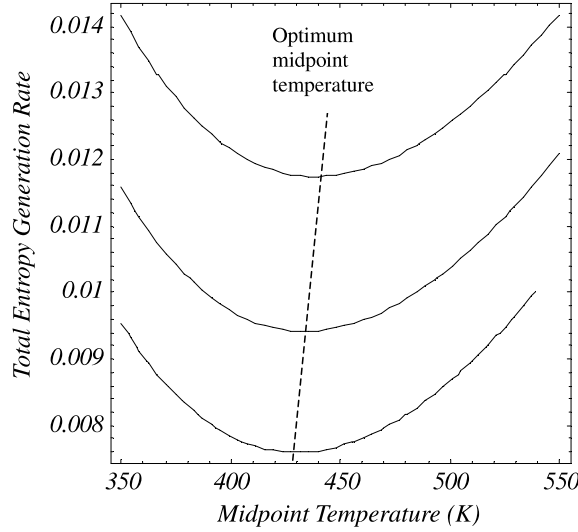


Fig. 5. Total entropy generation rate in thermoleg for a given thermoelectric figure of merit (optimum location is indicated).

The preceding equations are somewhat lengthy, and Fig. 5 shows the entropy generation rate along a thermoelectric leg of unit length for three values of the midpoint temperature.

Integrating over the two hot and cold segments and summing gives the total internal entropy generation rate for the whole leg as

$$\dot{S}_{\text{gen,int}} = \frac{2kA}{L} \left(\frac{T_H}{T_m} + \frac{T_m}{T_H} + \frac{T_m}{T_L} + \frac{T_L}{T_m} - 4 \right) + \frac{1}{4} I^2 \frac{\rho L}{A} \left(\frac{1}{T_H} + \frac{1}{T_L} + \frac{2}{T_m} \right) \quad (44)$$

Differentiating with respect to T_m and setting to zero gives:

$$T_{m,\text{opt}} = \sqrt{T_H T_L \left(1 + I^2 \frac{\rho}{k} \left(\frac{L}{A} \right)^2 \frac{1}{8\bar{T}} \right)} \quad (45)$$

which is identical to Eq. (16). This can be written in terms of the thermoelectric figure of merit (for a load ratio of unity):

$$T_{m,\text{opt}} = \sqrt{T_H T_L \left(1 + \frac{1}{32} \frac{Z \Delta T^2}{\bar{T}} \right)} \quad (46)$$

This shows that the optimum internal temperature that minimizes the internal entropy generation rate is very similar to the NCCA condition. When $Z = 0$ (no thermoelectric effect), the NCCA condition is obtained. Now using Eq. (41) with $m = 1$, it is seen that:

$$T_{m,\text{opt}} = \sqrt{T_H T_L \frac{T_m}{\bar{T}}} \quad (47)$$

Thus, when the midpoint temperature is the mean temperature, the NCCA case is obtained. However, with thermoelectricity present to a significant extent (by proper semiconductor material selection), the midpoint temperature is greater than the mean temperature.

The optimum (minimum) entropy generation rate of the altered system in terms of the original (prior to midpoint temperature alteration) is

$$\dot{S}_{\text{gen,int,min}} = 4k \frac{A}{L} \sqrt{\frac{\bar{T}^2 T_m}{T_H T_L}} \left\{ \frac{\sqrt{\bar{T}}}{T_m} + 1 \right\} + \frac{1}{8} \frac{\alpha^2}{\rho} \frac{A}{L} (T_H - T_L)^2 \frac{\bar{T}}{T_H T_L} \left\{ \sqrt{\frac{T_H T_L}{\bar{T} T_m}} + 1 \right\} \quad (48)$$

where $m = 1$ has been used.

Approximating the optimum by the NCCA condition, the minimum internal entropy generation rate becomes:

$$\dot{S}_{\text{gen,int,min}} \approx \frac{1}{2} I^2 \rho \frac{L}{A} \left(\frac{\bar{T} + \sqrt{T_H T_L}}{T_H T_L} \right) + 4k \frac{A}{L} \left\{ \frac{\bar{T} \sqrt{T_H T_L}}{T_H T_L} - 1 \right\} \quad (49)$$

or

$$\dot{S}_{\text{gen,min}}^* = \dot{S}_{\text{gen,min}} / kA/L \approx \frac{1}{8} Z \bar{T} \frac{\left(1 + \frac{\sqrt{T_H T_L}}{\bar{T}} \right)}{T_H T_L} + 8 \left(\frac{\bar{T} \sqrt{T_H T_L}}{T_H T_L} - 1 \right) \quad (50)$$

where it has been assumed that the NCCA optimum applies. In fact, as Z gets larger, the optimum intermediate temperature gets increasingly different, and Eqs. (49) and (50) become less accurate. For current and near future materials, Z is apparently limited to be below 0.01 (the ceiling on ZT is considered to be in the range 2–4 in the usual temperature range [12]). The optimum intermediate temperature, as a result, is very close to the NCCA temperature and will be within 5–10% of it.

9.2. Power evolution in a temperature gap including thermoelectric effect

The basis is a temperature gap (insulation) of fixed geometry and properties. Allowing for thermoelectric effects, the irreversibilities are clearly due to conduction and Joule heat, as given by Eqs. (29) and (30). The power generated in a gap with the aforementioned irreversibilities is found by considering that the steady state Guoy–Stodola theorem applies (whether this is the case is another question). For a gap with power recovery anticipated, the appropriate heat flows are $\dot{Q}_H$ and $\dot{Q}_L$ entering and leaving the gap respectively. Considering that no external irreversibilities exist, entropy is generated only internally as $\dot{Q}_H$ and $\dot{Q}_L$, crossing at T_H and T_L , respectively, traverse the gap. Thus, the power is equal to the maximum available power minus the destroyed availability due the irreversibilities present. This can be written as

$$\dot{W} = \eta_c \dot{Q}_H - T_L \dot{S}_{\text{gen},0} \quad (51)$$

Here, $S_{\text{gen},0}$ is simply the entropy generated within the material (for external irreversibilities Eq. (7) may be referred to). Since $\dot{S}_{\text{gen},0} = -(\dot{Q}_H/T_H) + (\dot{Q}_L/T_L)$, the above equation can be clearly seen to be the same as, $\dot{W} = \dot{Q}_H - \dot{Q}_L$, which is obtainable from a simple energy balance.

Now, if the midpoint temperature is allowed to vary somehow, the power in this case will be equal to the original power (with the midpoint at its natural value) less the added destruction of availability due to the alteration. Hence:

$$\dot{W}' = \eta_c \dot{Q}_H - T_L \dot{S}_{\text{gen},m} \quad (52)$$

where $S_{\text{gen},m}$ is the entropy generated including midpoint temperature variation and Q_H is evaluated using Eq. (26) at the hot side. Using Eq. (44) and eliminating the current, this is written as

$$\dot{W}' = \frac{1}{16} \frac{\alpha^2}{\rho} \frac{A}{L} (T_H - T_L)^2 \left(\frac{T_L}{T_H} - 2 \frac{T_L}{T_m} + 5 \right) - \frac{kA}{L} T_L \left(\frac{T_H}{T_L} - 2 \frac{T_m}{T_L} + 1 \right) \left(2 \frac{T_L}{T_m} - \frac{T_L}{T_H} - 1 \right) \quad (53)$$

This equation is to be compared with the well known equation for thermoelectric power at matched load condition (for a single leg):

$$\dot{W} = \frac{1}{4} \frac{\alpha^2}{\rho} \frac{A}{L} (T_H - T_L)^2 \quad (54)$$

The power factor (α^2/ρ) is assumed known for a given material at the average temperature, and the geometry (A/L) is fixed. Thus, the power generation can be found as a function of the midpoint temperature. The original midpoint temperature is found from Eq. (28) as

$$T_{m0} = \bar{T} + \frac{1}{8} I^2 \frac{\rho}{k} \left(\frac{L}{A} \right)^2 = \bar{T} + \frac{1}{32} Z \Delta T^2 \quad (55)$$

Fig. 6 shows the power output versus the midpoint temperature of a leg for a given temperature gap of 300 K. The power demonstrates an optimization as the midpoint temperature is decreased below its natural thermoelectric value (455 K) and below the mean temperature (450 K). As the temperature difference grows, the power increases, as shown in Fig. 7. The benefit from midpoint temperature decrease falls as the average temperature of the leg increases, as shown also in the last figure.

In the previous calculations, realistic values of material and geometrical parameters have been used. Thus, in the case of a ΔT of 300 K, a power per single leg of 0.42 W in the normal case (using

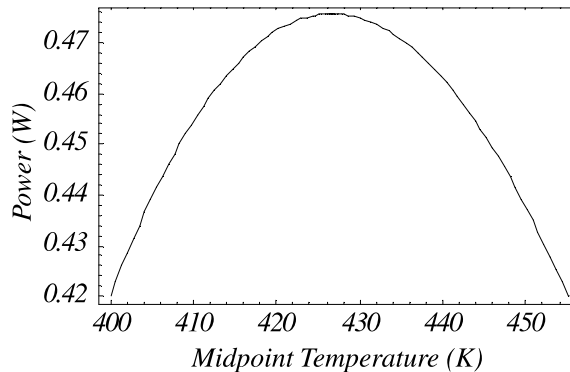


Fig. 6. Power versus midpoint temperature in a thermoleg with given geometry, material properties and imposed temperature difference.

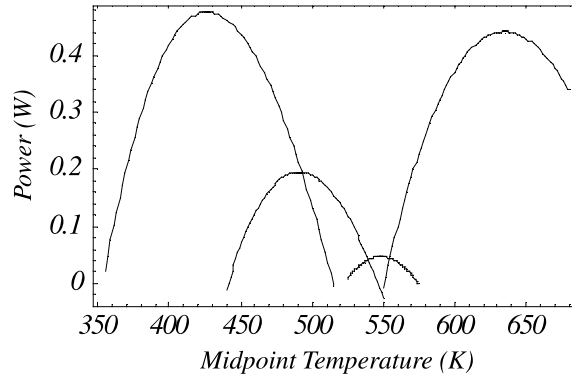


Fig. 7. Power versus midpoint temperature in a thermogleg for different temperature difference's and average temperatures.

Eq. (54)) and over 0.47 W in the optimized case is realized. Considering that in a module, there are usually multi-couples, the benefit can be significant.

9.3. Obtaining more power

When power is extracted, the heat flows in and out of the thermogleg are given by the two Eq. (39) evaluated at $x = 0$ and $x = L$, respectively. The power obtainable is still given by Eq. (51), except that the heat flows are now $\dot{Q}'_H$ and $\dot{Q}'_L$. Thus, the power is the difference: $\dot{W}' = \dot{Q}'_H - \dot{Q}'_L$. This gives, for any load ratio, in terms of the average and midpoint temperature:

$$\dot{W}' = \frac{2m+1}{2(1+m)^2} \frac{\alpha^2 A}{\rho L} (T_H - T_L)^2 + 4 \frac{kA}{L} T_L (\bar{T} - T_m) \quad (56)$$

So that at matched load and by eliminating the average temperature using Eq. (41) in favor of the original midpoint temperature, the maximum power is

$$\dot{W}'_{\max} = \frac{1}{4} \frac{\alpha^2 A}{\rho L} (T_H - T_L)^2 + 4 \frac{kA}{L} T_L (T_{m0} - T_m) \quad (57)$$

Clearly, with the original midpoint temperature, the normal thermoelectric power equation is obtained. For a midpoint temperature less than the original, more power is obtained. At a midpoint temperature equal to the average of the thermal gap, 12.5% more power is extracted (holding all else constant).

The previous results indicate that cooling the midpoint enhances the power production from a thermogleg. The cooling is assumed to be obtained independently, and it is not questioned here whether such a “waste-cooling” stream could be envisaged, say by thermal superconducting!

In order to ascertain if it may be possible to obtain a benefit from midpoint cooling, a net available power (exergy) assessment must be made. Such an assessment is not quite straightforward, since the device being analyzed is inherently irreversible. An interesting question is the effect of having different temperature profiles in each leg. One simple practical example is when one leg

is “passive”, that is, it does not contribute to any significant extent to the Seebeck coefficient. In such a case, the temperature profile in such a leg could be considered linear, being simply due to pure heat conduction. The cooling of the thermoleg midpoint may be considered to be equivalent to a two-stage cascade, although there are some differences.

9.4. Cascading

A simple two-stage thermoelectric cascade calculation will show that while the efficiency of the cascade increases, the total power is less than twice the power obtainable from the a single stage that utilized the full temperature difference available. In fact, power rises as the leg length is reduced (at least up to a certain minimum length in realistic cases that include contact resistances).

Consider a two-stage cascade of total length L with each stage having a length $L/2$. The cascade is within a total temperature difference of $\Delta T = T_H - T_L$ and each stage has the same cross-sectional area and material properties. The total power is given by

$$\dot{W}_T = \dot{W}_1 + \dot{W}_2 = \frac{1}{2} \frac{\alpha^2}{\rho} \{ \Delta T^2 + 2T_1(T_1 - 2\bar{T}) \} \quad (58)$$

where T_1 is the intermediate (single) temperature of the common interface of the two stages. Clearly, when $T_1 = \bar{T}$, the total power would be equivalent to that obtainable from a single stage case with a length of L and subject to the same overall ΔT . If the intermediate temperature is either less than or greater than the mean, more power can be obtained than in the previous case. This is clearly due to the diminishing of the temperature difference in either the upper or lower cascade. Hence if $T_1 = T_L$, the lower stage is simply a passive thermal superconductor and the power is due to the top stage of half the length operating with the full original temperature difference.

A difference between a two-stage cascade and a midpoint-cooled case is that the latter is actually a single electrical stage with one internal current. Additionally, in a cascade, there are always contact resistances present. As a possible example of tailoring the temperature profile, segmented thermoelements may be considered. In general, the internal temperature approach followed in this paper is somewhat different and sheds more light on the ongoing desire to improve thermoelectric performance.

10. Concluding remarks

The entropy generation rate and power production in a power producing system taken as a temperature gap between two thermal reservoirs was studied first in order to clarify the thermodynamic modeling of such systems. The influence of the presence of an electric field was considered. Thermoelectricity, as a case when an electric field is present as a result of the thermal field, was considered. Among several results, it was shown that there exists a possibility to enhance thermoelectric performance by tailoring the temperature profile. No attempt was made to suggest practical means of achieving the shown results.

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