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Effect of heat leak on cascaded heat engines

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Abstract

The effect of heat leak on the performance of cascaded heat engines is investigated. Two heat leak models are suggested. In the first model, the heat leak occurs across consecutive engines. In the second model, the heat leak takes place between the first engine and all other engines. Analytical expressions for the overall efficiency and overall output power of N cascaded engines in terms of the individual efficiencies of the engines, leakage fractions and heat input are derived for both models. The effect of cascading is found to increase the overall efficiency and power output, while the effect of heat leak is to diminish this rate of increase. Optimization of performance is possible for the first model only. Analytical expressions for the intermediate temperatures that maximize power and efficiency in addition to formulae for maximum power and efficiency, for the case of equal heat leak conductances, are developed. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Heat engine; Cascading; Heat leak; Entropy; Generation minimization; Finite time thermodynamics; Power maximization

1. Introduction

Modeling of complicated thermal systems using the heat engine concept facilitates their study and allows tuning of their physical characteristics in order to optimize their performance. In fact, thermodynamic optimization of heat engines or refrigerators has been the subject of many studies and has led to what has become an independent discipline denoted in the literature by either the entropy generation minimization technique (EGM) [1] or finite time/size thermodynamics (FTT) [2].

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Nomenclature

K, K_1, K_2	heat transfer coefficients (W/K)
$\dot{W}$	power output (W)
$\dot{Q}_L$	rate of heat entering heat sink (W)
$\dot{Q}_H$	rate of heat leaving heat source (W)
T_L	temperature of heat sink (K)
T_H	temperature of heat source (K)
T_i	temperature of hot side ith internal stage
η	efficiency of heat engine
η_C	Carnot efficiency
η_{max}	maximum efficiency
η_{mp}	efficiency at maximum power
τ	overall temperature ratio
τ_1	sink temperature to intermediate temperature ratio for two stage heat engine
f	heat leak fraction with equal conductances $[K(T_H - T_L)/\dot{Q}_H]$
f_1	a form of heat leak fraction based on initial stage conductance $(K_1 T_H/\dot{Q}_H)$

Most of the reported studies [3–14] have dealt with optimizing the performance of thermal systems, which can be represented by a single endoreversible heat engine, through maximizing either efficiency or power. Gordon and Zarmi [3], DeVos and Flater [4] and Nuwayhid and Moukalled [5] modeled the earth and its envelope using a Carnot-like engine and reported on the conversion efficiency of solar energy into wind energy [3,4] and the effect of the planet thermal conductance on this efficiency [5]. Nulton et al. [6] and Pathria et al. [7] described a set of feasible operations of a heat engine subject only to thermal losses in terms of an inequality similar to the second law of thermodynamics and applied it to Carnot-like refrigerators and heat pumps. Moukalled [8] and Moukalled and Nuwayhid [9] expanded the applicability and usefulness of the Curzon–Ahlborn concept [10] and made it more realistic by adding a heat leak term into a variation of a model developed by DeVos [11]. Gordon [12], Nuwayhid and Moukalled [13] and Nuwayhid et al. [14] modeled a thermoelectric generator as a heat engine and analyzed its performance.

On the other hand, work on cascaded heat engines has been rather limited [15–17] despite the fact that cascading has long been recognized as a way of increasing efficiency. In addition, the models used in the analysis generally neglected heat leak. Through the use of the power maximization method, Bejan [15] optimized the heat exchange allocation of a combined cycle power plant. El Haj Assad [16] employed the same technique to maximize the output power of two heat engines connected in series. Several workers have optimized the performance of multi-component cycles (e.g. Göktun [17], who optimized the performance of a combined absorption and ejector refrigerator).

The introduction of a bypass heat leak as a source of irreversibility gives a more realistic view of performance. The effect of heat leak on the performance of a single heat engine has been studied in several papers in the literature (see Ref. [18]). Bejan [1] summarized the heat leak treatment in several references and differentiated clearly between the bypass heat leaks, the concern of this paper, and other heat leak definitions. Moreover, the analysis of several heat engines connected in series has been confined to a definite number of engines governing a specific application. To the best of the author's knowledge, no work on studying the performance of N cascaded engines with heat leak has been reported. It is the intention of this work to undertake this task. In fact, the modeling of the heat leak itself is by no means clear or definite, and several heat leak scenarios are possible: across individual engines or from the first to each following engine. This paper reports on the performance of N cascaded heat engines in the presence of either of the aforementioned two types of heat leak. When studying multi-stage heat engines, a major concern related to a clear definition of the thermodynamic system arises. Thus, system–surroundings interactions and the relation of one stage to the other and the rest of the universe must be addressed. System–surroundings heat interactions, which tend to decrease further the efficiency of the cascaded engines, are not considered at this stage. This would be a work in progress, and the only source of irreversibility in the model considered herein is confined to the bypass heat leak term. It must be pointed out that the heat leak being studied here is an "internal" bypass heat leak and not a "work leak", such as may be due to friction or Joule heat. Thus, the bypass heat leak is, in fact, due to the thermal resistance of the power plant, which is sitting within a certain temperature range. In particular, the system may be defined as the working fluid of the heat engine. The heat leak discussed in this paper is the bypass heat from the hot part of the cycle to the cold part. A look ahead at (Fig. 1) shows the system within the "dotted line".

In the remainder of this article, the decrease in efficiency in the presence of heat leak is first highlighted for a single heat engine. Then, the increase in efficiency with cascading is demonstrated for the case of N engines with no heat leak. This is followed by a detailed study of the effect of heat leak on the performance of N cascaded heat engines for the two heat leak models described above. The paper ends with a section on the optimization of performance of N cascaded heat engines in the presence of heat leak.

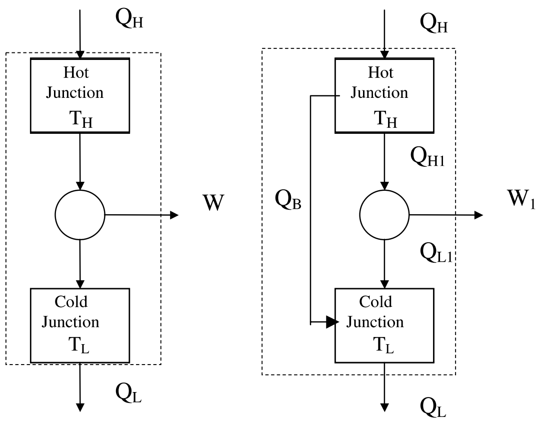

Fig. 1. Schematic of the externally reversible heat engine model (a) without and (b) with heat leak.

2. The effect of heat leak

Consider a heat engine with heat input rate $\dot{Q}_{\mathrm{H}}$ and heat output rate $\dot{Q}_{\mathrm{L}}$ operating between heat source temperature T_{H} and heat sink temperature T_{L}. The power producing compartment is reversible (a Carnot cycle), and thus, the engine is so-called "endoreversible". The heat input, heat output, power and efficiency (Fig. 1(a)) are given by $\dot{Q}_{\mathrm{H}}$, $\dot{Q}_{\mathrm{L}}$, $\dot{W}$ and η, respectively. With a bypass heat leak (Q_{B}) incorporated as shown in Fig. 1(b), the model per stage emerges as: "A Carnot engine operating in parallel with a bypass heat leak".

The bypass heat leak is taken to include, in a simplistic manner, the properties of the system. It is easily shown that the efficiency with heat leak (η_{B}) is related to that with no heat leak (η) via the following equation:

$$\eta_{\mathrm{B}} = (1 - L)\eta, \tag{1}$$

where L is the ratio of heat leak to heat input ($\dot{Q}_{\mathrm{B}}/\dot{Q}_{\mathrm{H}}$). The presence of the heat leak clearly reduces the efficiency in a linear fashion. This is a result of the heat leak competing with the Carnot component of the engine for the available heat input. With no bypass heat leak ($L = 0$), the efficiency reverts to the usual case.

3. Cascaded heat engines with no heat leak

When considering more than one heat engine stage, the subsequent stages constitute separate systems (or may be the surroundings of stage one). Thus, there is entropy generation (or equivalent interfacing mechanisms) between the stages. While somewhat unrealistic, in this work, it is assumed that neglect of the inter-stage resistance to heat flow is somehow taken into account in the bypass leakage term of the subsequent stage (see Fig. 2(a) and (b)).

With the above taking into account in general, it is quite easy to show that the efficiency of a cascaded heat engine with N stages can be written as:

$$\eta = 1 - \prod_{i=1}^{N}(1 - \eta_i), \tag{2}$$

where η_i is the efficiency of the ith stage, defined as the ratio of the work output from the stage divided by the heat input to the stage. The efficiency is clearly improved considerably by cascading. This is practically applied in the so-called topping or bottoming cycles.

4. Heat leak across individual engines model

A schematic of the model is depicted in Fig. 3. As shown, the heat leak is assumed to take place from the hot to the cold reservoir of each heat engine (or stage). The effect of the heat leaks on the overall efficiency of the cascaded engine configuration is studied by first considering a two and then a three engine scheme. The results are then generalized to a cascade of N engines connected in series, with the heat rejected from the ith engine being the heat input to the (ith + 1) engine.

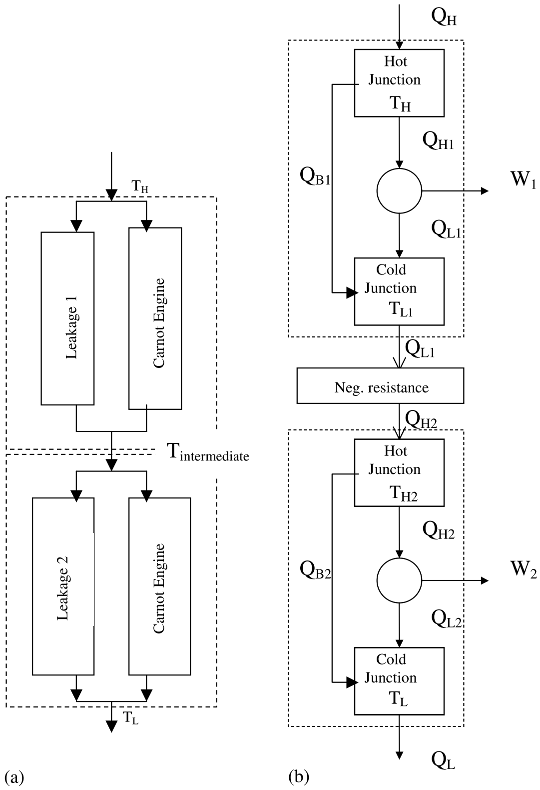

Fig. 2. Schematic of the two stage heat engine model: (a) basic model and (b) working model.

4.1. Two stage engine with heat leak

While giving a clear indication of the trend, the efficiency obtained by the previous model for cascaded heat engines can be made more realistic by including heat leak prescriptions. In fact, by simple first law considerations, the effect of a heat leak can be generalized. For the two stage heat engine shown in (Fig. 4), the power and efficiency relations per stage are as follows:

$$\begin{aligned}
\dot{W}_1 &= \dot{Q}_{\mathrm{H1}} - \dot{Q}_{\mathrm{L1}}, \\
\dot{W}_2 &= \dot{Q}_{\mathrm{H2}} - \dot{Q}_{\mathrm{L2}}, \\
\eta_1 &= \frac{\dot{W}_1}{\dot{Q}_{\mathrm{H1}}}, \\
\eta_2 &= \frac{\dot{W}_2}{\dot{Q}_{\mathrm{H2}}}.
\end{aligned} \tag{3}$$

The combined efficiency is then defined as:

$$\eta = \frac{\dot{W}_1 + \dot{W}_2}{\dot{Q}_{\mathrm{H}}}. \tag{4}$$

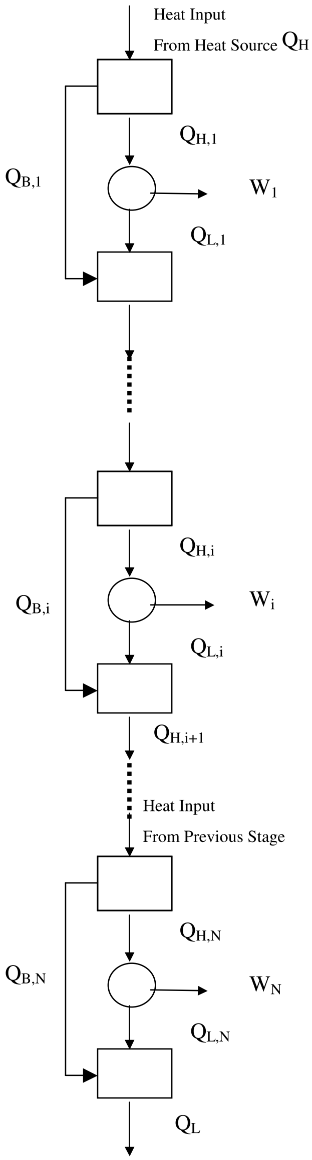

Fig. 3. *N* stages cascaded heat engine with heat flows shown.

After some manipulation involving the elimination of $\dot{Q}_{L1}$ and $\dot{Q}_{L2}$, the following is obtained for the combined efficiency

$$\eta = [1 - (1 - \eta_1)(1 - \eta_2)] - L_1\eta_1(1 - \eta_2) - L_2\eta_2, \tag{5}$$

where $L_1 = \dot{Q}_{B1}/\dot{Q}_H$ and $L_2 = \dot{Q}_{B2}/\dot{Q}_H$ are the partial heat leak ratios in each stage.

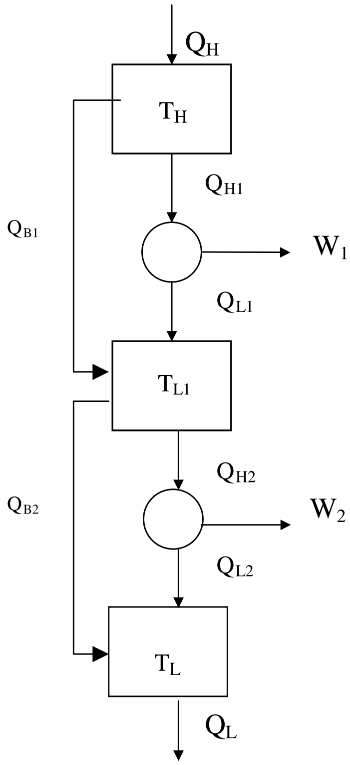

Fig. 4. Schematic of a two stage heat engine with partial heat leaks across each stage.

4.2. Three stage engine with heat leak

Based on Eq. (4), the overall efficiency of a three stage engine, (Fig. 5), is the sum of the two stage case with an additional contribution from the third stage. Thus:

$$\eta = \eta_{\text{two stage}} + \eta_{\text{additional}}. \tag{6}$$

Moreover, the power generated by stage three can easily be shown to be:

$$\dot{W}_3 = \eta_3 \lfloor (1-\eta_1)(1-\eta_2)(\dot{Q}_{\mathrm{H}} - \dot{Q}_{\mathrm{B1}}) + (1-\eta_2)(\dot{Q}_{\mathrm{B1}} - \dot{Q}_{\mathrm{B2}}) - (\dot{Q}_{\mathrm{B2}} - \dot{Q}_{\mathrm{B3}}) \rfloor. \tag{7}$$

So that, for a three stage engine with three partial heat leaks, the overall efficiency is:

$$\eta = [1-(1-\eta_1)(1-\eta_2)(1-\eta_3)] - L_1\eta_1(1-\eta_2)(1-\eta_3) - L_2\eta_2(1-\eta_3) - L_3\eta_3, \tag{8}$$

where the Ls are the partial heat leak ratio’s (from one stage to the following).

4.3. Generalizing the effect of heat leak on cascaded heat engines

The two previous cases appear to show a trend and suggest the following overall efficiency for N cascaded heat engines, each with a separate heat leak:

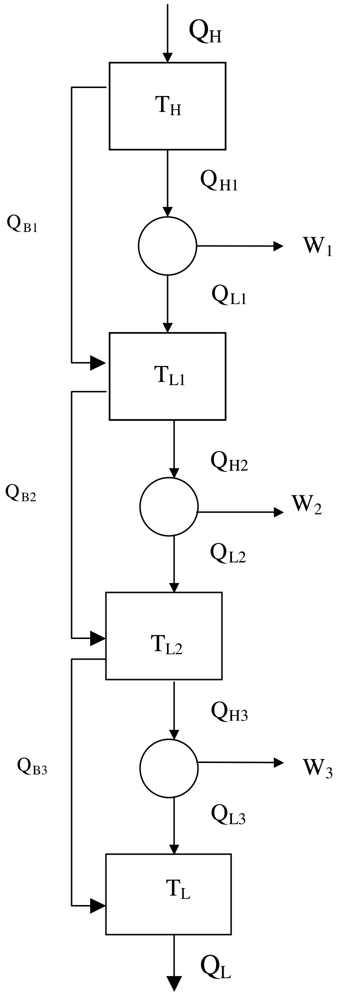

Fig. 5. Schematic of a three stage heat engine with partial heat leaks across each stage.

$$\eta = 1 - \prod_{i=1}^{N}(1-\eta_i) - \sum_{i=1}^{N}\left[L_i\eta_i\prod_{j=i}^{N}(1-\eta_{j+1})\right], \quad \eta_{N+1} = 0, \tag{9}$$

where L_i is the partial heat leak across stage i. The last term in the above equation is obviously the contribution of all N heat leaks.

The proof to the above equation is done by induction. For that purpose, Eq. (9) is assumed to be valid for $N-1$ cascaded engines and shown to hold for N cascaded engines.

As mentioned in the previous subsection, the overall efficiency of N cascaded engines is the sum of the overall efficiency of the $N-1$ cascaded engines and the contribution of the Nth engine. Therefore,

$$\eta_N = \eta_{N-1} + \frac{\dot{W}_N}{\dot{Q}_{\mathrm{H}}} = \eta_{N-1} + \eta_N\frac{\dot{Q}_{\mathrm{H},N}}{\dot{Q}_{\mathrm{H}}}. \tag{10}$$

From (Fig. 3), it is seen that:

$$\dot{Q}_{\mathrm{H},N} = \dot{Q}_{\mathrm{Rejected\ at}\ N-1} - \dot{Q}_{\mathrm{B},N} \tag{11}$$

and using the following fact,

$$\dot{Q}_{\mathrm{Rejected\ at}\ N-1} = (1-\eta_{N-1})\dot{Q}_{\mathrm{H}}, \tag{12}$$

the overall efficiency of N cascaded heat engines with heat leak is found to be:

$$\eta_N = \eta_{N-1} + \eta_N \frac{(1-\eta_{N-1})\dot{Q}_{\mathrm{H}} - \dot{Q}_{\mathrm{B},N}}{\dot{Q}_{\mathrm{H}}}, \tag{13}$$

leading to:

$$\eta_N = 1 - (1-\eta_{N-1})(1-\eta_N) - \eta_N L_N, \tag{14}$$

where L_N is the heat leak ratio of the Nth (last) engine. Substituting the value of η_{N-1} into Eq. (14), as obtained from Eq. (9), and rearranging gives:

$$\eta_N = 1 - (1-\eta_N)\prod_{i=1}^{N-1}(1-\eta_i) - (1-\eta_N)\sum_{i=1}^{N-1}\left[L_i\eta_i\prod_{j=i}^{N-1}(1-\eta_{j+1})\right], \quad \eta_{N+1} = 0 \tag{15}$$

which is obviously Eq. (9). Eq. (9) is, thus, considered to be generally valid.

For the special case when the heat leak fractions (L_i) are equal for all stages, the overall efficiency turns out to be the same as for N cascaded engines with heat leak from the first to the final stage only (Fig. 6). For example, for two or three stage cases, the overall efficiency with the same single step leak (L) is:

$$\begin{aligned} \eta &= (1-L)[1-(1-\eta_1)(1-\eta_2)], \\ \eta &= (1-L)[1-(1-\eta_1)(1-\eta_2)(1-\eta_3)] \end{aligned} \tag{16}$$

and in general:

$$\eta = (1-L)\left[1 - \prod_{i=1}^{N}(1-\eta_i)\right] \tag{17}$$

which is consistent with Eq. (1). To explain this behavior, it is noticed that, intuitively, the heat leak fraction should decrease while passing from one engine to the following (since Q_{H} is constant and work is being extracted). Forcing the heat leak fraction to be constant, is similar to assuming that it is passing passively from one stage to the next without having any effect on the intermediate stages, which is similar to a model with heat leak between the first and the last stage only.

5. Heat leak from the first to each following engine model

A schematic of the model is depicted in (Fig. 7). As shown, the heat leak is assumed to take place from the hot reservoir of the first engine to all other heat reservoirs that are at lower temperatures. As was done with the previous model, the effect of the heat leak on the overall efficiency of the cascaded engine configuration is studied by first considering a two and then a

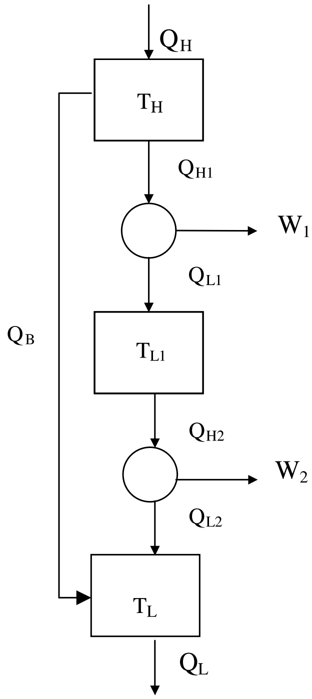

Fig. 6. Schematic of a two stage heat engine with a direct (overall) single step heat leak across it.

three engine scheme. The results are then generalized to a cascade of N engines. Then, the generalized equation is proved by using the induction technique. For compactness (since the procedure is very similar to the previous one), only a summary of the intermediate steps is given.

By applying the same manipulations as before, it can be shown that for a two stage engine of the current model, the overall efficiency is:

$$\begin{aligned}\eta &= 1-(1-\eta_1)(1-\eta_2)-L_1\eta_1(1-\eta_2)-L_2[1-(1-\eta_1)(1-\eta_2)]\\ &= (1-L_2)[1-(1-\eta_1)(1-\eta_2)]-L_1\eta_1(1-\eta_2).\end{aligned} \tag{18}$$

This can be seen to be:

$$\eta = \eta_{\text{Overall},2} - L_1\eta_1(1-\eta_2), \tag{19}$$

where the first term on the right hand side of the above equation is the efficiency of a two stage engine with a single one step overall leakage from top to bottom. Considering more stages, the overall efficiency of N cascaded engines is found to be:

$$\eta = \left[1-\sum_{i=1}^{N} L_i\right]\left[1-\prod_{i=1}^{N}(1-\eta_i)\right] + \sum_{i=1}^{N-1} L_i\left[1-\prod_{j=i+1}^{N}(1-\eta_j)\right] \tag{20}$$

which, if multiplied out and expanded, yields a form similar to that of Eq. (18) that was given for the two stage case.

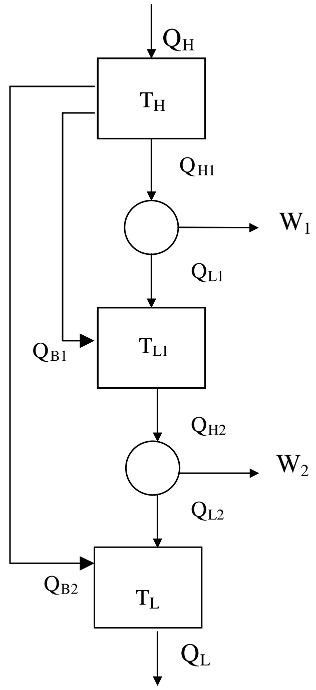

Fig. 7. Schematic of a two stage heat engine with heat leak from hot reservoir to cold side of each reservoir.

A more compact form lumping leakage terms separately (in conformity with Eq. (9)) is:

$$\eta = 1 - \prod_{i=1}^{N}(1-\eta_i) - \sum_{i=1}^{N}\left\{\left[L_i \prod_{j=i+1}^{N}(1-\eta_j)\right]\left[1-\prod_{k=1}^{i}(1-\eta_k)\right]\right\}, \quad \eta_{N+1} = 0. \tag{21}$$

This equation can, as was done with Eq. (9), be easily proven valid in general for any number of heat engines. Thus starting with the requirement that it be valid for $N-1$ engines, a statement similar to Eq. (10) can be written. The treatment given by Eq. (11) through Eq. (15) then leads to the desired generalized equation.

6. Numerical example

In order to appreciate the effect of the two heat leak scenarios, some numerical comparisons are performed. The numbers used are mostly based on current thermoelectric performance and are used because this type of generator suffers from relatively low efficiency and can benefit from cascading to increase its overall efficiency. However, a heat leak will reduce this increase, and it may be of interest to quantify such a decrease.

Thus, consider for simplicity and practicality a two stage generator: the first has an efficiency of 10%, while the second has an efficiency of 5%. The cascaded engine, not taking heat leaks into account, has a combined efficiency of 14.5%. Now, considering a one step heat leak across the generator, amounting to 10% of the heat input, the combined efficiency is reduced to 13.05%. A heat leak fraction of up to 31% will still provide an efficiency of 10% for the combined system.

Next, the heat leak is considered to occur in two stages (across the first generator and then across the second). If the partial heat leak fractions are the same (10%), the result is the same as the one step overall leak: for our case, an efficiency of 13.05% emerges. If, however, the first leak fraction is 10% and the second is 7.5% (arbitrarily chosen), the combined efficiency is 13.18%. One might be tempted to increase somewhat the first stage heat leak (by providing a thermal path) in order to increase the output of the second. Caution must be exercised, since they are intimately tied.

This example has clearly demonstrated how the heat leak decreases the overall efficiency of cascaded heat engines and thereby gives a more realistic picture of their performance.

7. Variation of the overall efficiency of a two stage heat engine

An interesting question to be answered is whether optimization of performance in the presence of heat leak is possible or not. For that purpose, the variation of efficiency with internal temperature ratio of a two stage engine is considered. As a first approximation for the purposes of the current work, the heat engines are taken to be endoreversible with efficiencies given by (refer to (Fig. 4)):

$$\eta_1 = 1 - T_{L1}/T_H \quad \text{and} \quad \eta_2 = 1 - T_L/T_{L1} \tag{22}$$

while the leakages are assumed to be of the convective/diffusive heat transfer mode (linear) and are given by:

$$L_1 = \frac{\dot{Q}_{B1}}{\dot{Q}_H} = \frac{K_1(T_H - T_{L1})}{\dot{Q}_H} \quad \text{and} \quad L_2 = \frac{\dot{Q}_{B2}}{\dot{Q}_H} = \frac{K_2(T_{L1} - T_L)}{\dot{Q}_H}, \tag{23}$$

where K_1 and K_2 are the thermal conductance's of the leakage paths.

Using the above in Eq. (5) gives the overall efficiency of a two stage engine with two heat leaks, one across each individual engine, as:

$$\eta = 1 - \tau - f_1\left[\frac{(\tau - \tau_1)^2}{\tau\tau_1} + \kappa\frac{\tau(1 - \tau_1)^2}{\tau_1}\right], \tag{24}$$

where $\tau = T_L/T_H$ is the temperature ratio, $\tau_1 = T_L/T_{L1}$ is the internal temperature ratio, $\kappa = K_2/K_1$ is the heat leak conductance ratio and $f_1 = K_1T_H/\dot{Q}_H$ is a form of leakage fraction. (Fig. 8) shows the overall efficiency of the two stage engine as a function of the internal temperature ratio (function of the internal temperature) for values of κ of 0.1, 0.5, and 0.9. The figure clearly demonstrates that optimization is warranted.

It should be pointed out here that the optimization of the second heat leak model leads only to the trivial solution $T_i = T_H$ where i is any intermediate temperature. This is due to the fact that the irreversibility is between any engine and the first engine and not between consecutive engines. Thus, the optimization presented in Section 8 will be confined to the first heat leak model only.

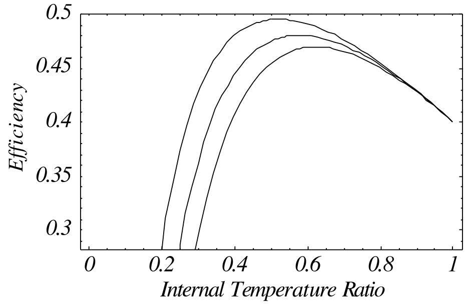

Fig. 8. Variation of overall efficiency of a two stage engine as a function of the internal temperature ratio for a temperature ratio $\tau = 0.5$ and leakage fraction $f_1 = 0.2$. The thermal conductance ratio (κ) increases from top to bottom as 0.1, 0.5 and 0.9.

8. Optimization

Having derived general expressions for the overall efficiency of cascaded heat engines, optimization of performance can be performed. For that purpose, two methods may be followed: the power maximization method (PM), or the EGM method. That the two methods are equivalent has been clearly elaborated in Ref. [1]. Additionally, Nuwayhid et al. [14] demonstrated the equivalence of the two methods for a thermoelectric generator.

In what follows, both techniques are used to optimize the performance of a two stage engine with heat leak across each individual engine and derive closed form optimum efficiency expressions. A generalized expression for the maximum overall efficiency of N cascaded engines with heat leak across each individual engine is only given for the case of equal heat conductances due to the fact that the general unequal conductance case results in a mathematically unmanageable optimum form.

8.1. Optimum intermediate temperature of a two stage engine using the PGM method

Since the heat input is fixed, the total power is obtained by multiplying the overall efficiency by Q_{H}. For the two stage engine with heat leak across each individual engine, the power is obtained from Eq. (24) and is given by:

$$\dot{W} = \frac{\dot{Q}_{\mathrm{H}} T_{1\mathrm{L}}(T_{\mathrm{H}} - T_{\mathrm{L}}) - K_1 T_{\mathrm{L}}(T_{\mathrm{H}} - T_{1\mathrm{L}})^2 + K_2 T_{\mathrm{H}}(T_{1\mathrm{L}} - T_{\mathrm{L}})^2}{\dot{Q}_{\mathrm{H}} T_{\mathrm{H}} T_{1\mathrm{L}}}. \tag{25}$$

Differentiating with respect to $T_{1\mathrm{L}}$ and setting equal to zero gives the intermediate temperature that optimizes power (also efficiency in this case) as:

$$T_{1\mathrm{L,opt}} = \sqrt{\frac{T_{\mathrm{H}} T_{\mathrm{L}}(K_1 T_{\mathrm{H}} + K_2 T_{\mathrm{L}})}{K_1 T_{\mathrm{L}} + K_2 T_{\mathrm{H}}}}. \tag{26}$$

That this value is bound by T_{H} and T_{L} can easily be verified by observing the following limits:

$$\lim_{K_1 \to \infty} T_{1\mathrm{L,opt}} = T_{\mathrm{H}} \quad \text{and} \quad \lim_{K_2 \to \infty} T_{1\mathrm{L,opt}} = T_{\mathrm{L}}. \tag{27}$$

8.2. Optimum intermediate temperature of a two stage engine using the EGM method

Since each stage is taken as endoreversible, the only sources of entropy generation (irreversibility) are the two heat leaks. The total entropy generation rate is:

$$\dot{S}_{\text{gen}} = K_1 \frac{(T_{\text{H}} - T_{1\text{L}})^2}{T_{\text{H}} T_{1\text{L}}} + K_2 \frac{(T_{1\text{L}} - T_{\text{L}})^2}{T_{1\text{L}} T_{\text{L}}}. \tag{28}$$

Differentiating with respect to $T_{1\text{L}}$ and setting to zero gives the same optimum intermediate temperature as reported above in Eq. (26). In this situation, it is actually easier to use the EGM method.

8.3. Optimum efficiency for a two stage engine

While the individual (internal) efficiencies of each stage are leakage dependent, the overall efficiency is only dependent on the heat input. It follows that the optimization of overall efficiency and overall power are the same. Using Eq. (26) for the optimum internal temperature, the optimum overall efficiency is obtained as:

$$\eta_{\text{opt}} = 1 - \frac{T_{\text{L}}}{T_{\text{H}}} - \frac{2}{Q_{\text{H}} T_{\text{H}}} \left[(K_1 T_{\text{L}} + K_2 T_{\text{H}}) \sqrt{\frac{T_{\text{H}} T_{\text{L}} (K_1 T_{\text{H}} + K_2 T_{\text{L}})}{K_1 T_{\text{L}} + K_2 T_{\text{H}}}} - (K_1 + K_2) T_{\text{H}} T_{\text{L}} \right]. \tag{29}$$

In the special case of equal conductances ($K_1 = K_2$), this reduces to:

$$\eta_{\text{opt}} = \overbrace{1 - \frac{T_{\text{L}}}{T_{\text{H}}}}^{=\eta_{\text{C}}} - \frac{2K\sqrt{T_{\text{H}} T_{\text{L}}}}{Q_{\text{H}} T_{\text{H}}} \left(\sqrt{T_{\text{H}}} - \sqrt{T_{\text{L}}} \right)^2 \tag{30}$$

or upon multiplying and dividing by $(T_{\text{H}} - T_{\text{L}})$ and introducing the temperature ratio $\tau = T_{\text{L}}/T_{\text{H}}$:

$$\eta_{\text{opt}} = 1 - \tau - \frac{4\tau}{1 - \tau} f \left(\frac{(1 + \tau)\sqrt{\tau}}{2\tau} - 1 \right), \tag{31}$$

where $f = K(T_{\text{H}} - T_{\text{L}})/\dot{Q}_{\text{H}}$ is a form of *heat-leak fraction*.

Eq. (30) shows that the efficiency is the Carnot efficiency reduced by an amount due to the incurred irreversibility. On the other hand, Eq. (31) shows that the efficiency (and power) drops as the heat leak fraction increases. Fig. 9 shows the optimized overall efficiency as a function of the temperature ratio for given heat leak ratios. It is apparent that for a given temperature ratio, the optimum overall efficiency is bound. Additionally, since the heat input is taken as fixed, the maximum power output is represented by any of the above two equations multiplied by $\dot{Q}_{\text{H}}$.

8.4. Optimum intermediate temperatures of N cascaded heat engines

The entropy generation rate can be written for as many stages as desired by summing the irreversibility contribution of each heat leak. Differentiation with respect to each intermediate temperature is performed and the resulting equation set to zero. The resulting system of equations, not presented here for compactness, has to be solved to obtain the various optimum

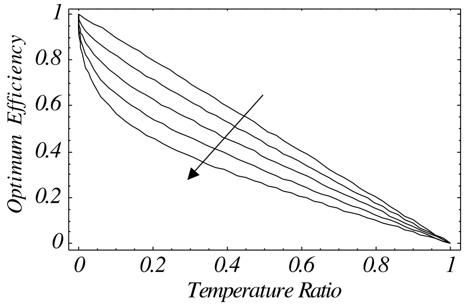

Fig. 9. Optimum efficiency of a two stage heat engine as a function of the temperature ratio (T_L/T_H). The heat leak fraction increases from top to bottom (as arrow indicates) as 0%, 25%, 50%, 75% and 100%.

intermediate temperatures. These equations are highly nonlinear and closed form solutions, if possible, are lengthy to present and hard to deal with. If the very reasonable assumption of equal conductances (i.e. $K = K_1 = K_2 = \cdots = K_N$) for each stage is used, the situation becomes quite manageable. Using this assumption, it is seen that for N cascaded heat engines, the intermediate temperatures that maximize the overall efficiency (or power) are given by:

$$T_{i,\mathrm{opt}} = T_{\mathrm{H}}^{(N-(i-1))/N} T_{\mathrm{L}}^{(i-1)/N}, \tag{32}$$

where i is the intermediate temperature number starting from $T_1 = T_H$ and ending with $T_{N+1} = T_L$. For a two stage engine, $T_{i,\mathrm{opt}} = (T_H T_L)^{1/2}$. The result is similar to the NCA [10] case of a heat engine optimized for maximum power by adjusting the hot side temperature (which is different from the heat source temperature due to the finite rate heat transfer). What is reported here is in relation to multi-stage heat engines and is, therefore, conceptually different from the NCA condition but can be thought of as a generalized form of the NCA relation for cascaded heat engines with heat leak.

8.5. Generalization of the optimum efficiency—case of equal conductances

The optimum efficiency for N cascaded heat engines with equal heat leak conductances is obtained by inserting the optimum intermediate temperatures (obtained from Eq. (32)) into the equation for efficiency (Eq. (9)). The result is:

$$\eta_{\mathrm{opt}} = \eta_{\mathrm{C}} - \frac{NK(T_{\mathrm{H}}T_{\mathrm{L}})^{(N-1)/N}}{Q_{\mathrm{H}}T_{\mathrm{H}}}\left(T_{\mathrm{H}}^{1/N} - T_{\mathrm{L}}^{1/N}\right)^2 \tag{33}$$

or in terms of temperature ratio and leakage fraction:

$$\eta_{\mathrm{opt}} = 1 - \tau - 2N\frac{\tau}{1-\tau} f\left(\frac{(1+\tau^{2/N})}{2\tau^{1/N}} - 1\right). \tag{34}$$

In the case of two stages, Eq. (30) is clearly reproduced. In (Fig. 10), the optimum efficiency of a cascaded engine is plotted as a function of the number of stages for a predetermined leakage fraction. The figure indicates that efficiency rises with number of stages and asymptotically approaches the Carnot limit. The efficiency is also clearly reduced with an increase in heat leak. It is

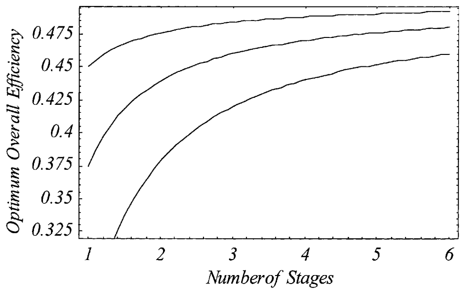

Fig. 10. Variation of the optimum overall efficiency with number of stages for equal conductances in each stage for $\tau = 0.5$ ($\eta_C = 0.5$). The leakage fraction (f) increases from top curve to bottom as 0.1, 0.25 and 0.5.

also apparent that for all practical purposes, going to more than a few stages makes little difference. The greater the heat leak, however, the more stages are required for a given efficiency.

9. Closing remarks

The effect of heat leak on the performance of cascaded heat engines was studied using two heat leak models, in which the heat leak occurred either across consecutive engines or between the first engine and all other engines. The basic model used in this work neglected the interaction between each engine and its surroundings (including the subsequent stages), resulting in a simplified analysis that nevertheless can be used in a preliminary assessment. A more realistic view addressing the interfacing between consecutive engines is to be addressed in a forthcoming paper. For both models conceived, general expressions for the overall efficiency and overall output power of N cascaded engines were derived. It was found that cascading increases the overall efficiency and power output, while the effect of heat leak is to diminish this rate of increase. Optimization of performance was possible for the case when the heat leak occurs between consecutive engines, and general maximum efficiency and power output equations were derived for the case of equal heat leak conductances.

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