

An Efficient Very High Resolution Scheme based on an Adaptive-Scheme Strategy

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Abstract

A new efficient Adaptive Very High Resolution (AVHR) scheme based on the Normalized Variable and Space Formulation (NVSF) methodology is presented in this paper. The scheme is composed of the High Resolution (HR) SMART scheme and a wide stencil scheme based on a bounded seventh order interpolation profile denoted here by BSEVENTH. The Normalized Weighting Factor (NWF) acceleration technique and the Deferred Correction procedure are used to implement the SMART and BSEVENTH schemes, respectively. The AVHR scheme switches to the seventh-order profile near a discontinuity or in the presence of a high gradient and to the SMART scheme in smooth regions. A new normalized switching criterion is developed and its accuracy illustrated for a number of scalar profiles. Numerical results for four purely convective test problems show that while preserving the accuracy of the BSEVENTH scheme, the new AVHR scheme reduces the cost by more than 400% if compared to the traditional DC implementation of the BSEVENTH scheme and by 50% when compared to a new accelerated NWF-DC implementation of the BSEVENTH scheme.

Nomenclature

A	Coefficients of the discretized equation.
Q	Source term in the conservation equation.
B	Source term integrated over one control volume in the discretized equation.
f()	Functional relationship.
J	Total scalar flux across cell face.
m	Slope in NVD.
ERROR	Residual error.
S	Surface of cell face.
u, v	Velocity components in the x- and y- directions.
ϕ	General dependent variable..
Γ	Diffusion coefficient.
ρ	Density.
ξ	Distance along coordinate line.
GRAD	Absolute gradient accross a cell face.
$\overline{\text{GRAD}}$	Normalized gradient across a cell face.

Superscripts

U	Upwind formulation.
D	Diffusion contribution.
C	Convection contribution.
~	Refers to normalized variable.

Subscripts

e,w,n,s	Refers to control volume faces.
E,W,N,S,...	Refers to neighbours of P grid point.
P	Main grid point.
f	Refers to a control volume face.
U	Upstream grid point.
D	Downstream grid point.
C	Central grid point.
nb	Refers to neighbours.
dc	Deferred correction.

Introduction

Important advances have been accomplished in the simulation of two-dimensional convection-diffusion transport problems. Errors caused by numerical diffusion and numerical dispersion have been largely reduced through the use of composite one-dimensional high resolution (HR) [1,2,3] and multi-dimensional very high resolution (VHR) [4,5] schemes. These have been formed by enforcing a monotonicity or convection boundedness criterion (CBC) [6] on different types of one-dimensional and multi-dimensional high order interpolation profiles. The use of one-dimensional high order profiles has largely decreased errors produced by streamwise numerical diffusion and to a lesser extent errors arising from cross-stream numerical diffusion. The utilization of multi-dimensional high order profiles however, has greatly reduced both components of numerical diffusion [4,7]. Moreover, enforcing a monotonicity criterion eliminated errors originating from numerical dispersion. The Total Variation Diminishing (TVD) is one framework for developing HR schemes [8]. Another more recent and elegant framework, is the Normalized Variable Formulation methodology (NVF) of Leonard [1] and its extension the Normalized Variable and Space Formulation methodology (NVSF) of Darwish and Moukalled [9]. Both frameworks have also been successfully used to bound the multi-dimensional streamline or skew schemes [4].

The next challenge to be addressed is the adaptation of the large number of HR and VHR schemes to three-dimensional simulations. The task appears to be a simple one, however, major difficulties related to the implementation and performance of these schemes in three dimensions arise. While schemes based on multi-dimensional profiles do not suffer with respect to accuracy in the move to three dimensions, higher implementation complexities combined with convergence and stability problems, increase appreciably their cost relative to locally one-dimensional schemes. On the other hand, implementation problems being negligible for schemes based on locally one-dimensional profiles, the higher dimensionality lowers their accuracy because of the increase in the cross-stream component of numerical diffusion. Thus, the widely used one-dimensional HR schemes based on third order interpolation profiles [10] become over diffusive in three-dimensional situations and do not produce results as accurate as those produced in two-dimensional applications. In order to improve predictions, schemes based on higher order interpolation profiles, fifth or even seventh order accurate, should be employed [11]. The use of these one-dimensional very high-order

interpolation profiles while relatively cheap for two-dimensional problems, become computationally very expensive when simulating three-dimensional flows, because of the larger computational stencils which, when combined with the Deferred Correction (DC) procedure [12], give rise to large source terms and lead to low convergence rates.

One way to address the above mentioned problems is through employing adaptive techniques; these can be divided into two groups that can be denoted as the adaptive grid and adaptive scheme methods. In the first technique, mesh points are concentrated only in areas where large discretization errors are observed. A large number of researchers have worked in this area and many local and global adaptive grid techniques have been developed over the years [13,14]. The majority of these techniques was developed using the finite element method (FEM) rather than the finite volume method (FVM) [14,15] because the use of irregular grids leads to a loss of efficiency in the coding and performance of algorithms based on the segregated approach. More recently, a Domain Decomposition approach [13,16] that gives substantial flexibility for FVM methods, has been developed. However problems pertaining to the transfer of information between grids and slow convergence rates are still plaguing this technique. In the second approach, the mesh is of a regular type and increased accuracy in regions of high gradients is obtained by switching adaptively to more accurate schemes [11,17]. Thus the expensive Very High Resolution (VHR) schemes are used only where needed while the cheaper HR schemes are used in the remaining computational regions. Few attempts based on the TVD approach to develop composite adaptive schemes have been reported [18]. Some work has also been done in the context of the NVF methodology [17], yet more work is still needed especially in devising proper switching criterion, and in assessing the efficiency of the adaptive approach followed.

In the present work, the second approach is selected. The difficulties involved in implementing adaptive schemes are investigated, and an adaptive scheme is developed and tested using the finite volume method. Moreover, a new criterion for selecting the order of the scheme to be used along a control volume face is devised. The adaptive scheme is based on the HR SMART scheme [19] combined with a bounded seventh order scheme [11,17]. The Adaptive Very High Resolution (AVHR) scheme is implemented, in the context of the NVF and NVSF methodologies, using a combination of the recently developed NWF acceleration technique [20] and the DC procedure [12].

For completeness, the discretization process for the general scalar equation is first presented. Then, the Convective Boundedness Criterion (CBC) is described in terms of the normalized variables approach along with the VHR scheme. This is followed by a detailed discussion of the newly devised selection criterion, adaptive strategy, and implementation of the adaptive scheme. Finally, a number of tests are carried out to measure the accuracy and computational efficiency of the newly developed scheme.

Numerical Discretization of the Transport Equation

The transport equation governing two dimensional incompressible steady flows may be expressed in the following general form:

$$\bar{\nabla} \cdot (\rho \bar{\mathbf{u}} \phi) = \bar{\nabla} \cdot (\Gamma \bar{\nabla} \phi) + Q \quad (1)$$

where ϕ is any dependent variable, $\bar{\mathbf{u}}$ is the velocity vector, and ρ , Γ , and Q are the density, diffusivity, and source term respectively. Integrating the above equation over the control volume shown in Figure 1(a), and applying the divergence theorem, the following discretized equation is obtained:

$$J_e + J_w + J_n + J_s = B \quad (2)$$

where J_f represents the total flux of ϕ across cell face 'f' (f= e, w, n or s), and B is the volume integral of the source term Q . Each of the surface fluxes J_f contains a convective contribution, J_f^C , and a diffusive contribution, J_f^D , hence:

$$J_f = J_f^C + J_f^D \quad (3)$$

where the diffusive flux is given by:

$$J_f^D = (-\Gamma \nabla \phi)_f \cdot \mathbf{S}_f \quad (4)$$

and the convective flux by:

$$J_f^C = (\rho \bar{\mathbf{u}} \cdot \mathbf{S})_f \phi_f = C_f \phi_f \quad (5)$$

where $\mathbf{S}_f$ is the surface of cell face 'f', and C_f is the convective flux coefficient at cell face 'f'. The diffusive flux at the control volume face 'f' is discretized using a linear symmetric interpolation profile so as to write the gradient as a function of the neighboring grid points.

As can be seen from Eq. (5), the accuracy of the control volume solution for the convective scalar flux depends on the proper estimation of the face value (ϕ_f) as a function of the neighboring ϕ node

values. Using some assumed interpolation profile, ϕ_f can be explicitly formulated by a functional relationship of the form:

$$\phi_f = f(\phi_{NB}, C_f) \quad (6)$$

where ϕ_{NB} denotes the neighboring ϕ node values ($\phi_E, \phi_W, \phi_N, \phi_S, \phi_P, \phi_{EE}, \phi_{WW}, \phi_{NN}, \phi_{SS}$, etc...). The interpolation profile may be one-dimensional or multi-dimensional of low or high order of accuracy. The higher the order of the profile is, the lower numerical diffusion will be. However, the order of the profile and its dimensionality do not reduce numerical dispersion.

After substituting the face values by their functional relationships relating to the node values of ϕ , Eq. (2) is transformed after some algebraic manipulations into the following discretized equation:

$$A_P \phi_P = \sum_{NB} A_{NB} \phi_{NB} + B_P \quad (7)$$

where the coefficients A_P and A_{NB} depend on the selected scheme and B_P is the source term of the discretized equation. An equation similar to Eq. (7) is obtained at each grid point in the domain and the collection of all these equations form a system of algebraic equations that is solved here iteratively to obtain the ϕ field.

Normalized Variables and CBC criterion

As mentioned above, increasing the order and/or dimensionality of the interpolation profile do not reduce errors caused by numerical dispersion. To eliminate this error, limiters on the convective flux should be imposed. The flux limiter denoted by the Convective Boundedness Criterion (CBC) is adopted here and explained next in terms of the normalized variables approach. Figure 1(b) shows the local behavior of the convected variable near a control-volume face. If the value of the dependent variable at the control volume face located at a distance ξ_f from the origin is expressed by ϕ_f , then the normalized variables (Figure 1(c)) will be defined as [9,21]:

$$\begin{aligned} \tilde{\phi} &= \frac{\phi - \phi_U}{\phi_D - \phi_U} \\ \tilde{\xi} &= \frac{\xi - \xi_U}{\xi_D - \xi_U} \end{aligned} \quad (8)$$

where the subscripts U and D that depend on the flow direction, refer to upstream and downstream locations, respectively.

Using the normalized variables the convection boundedness criterion for implicit steady state flow calculation [6], states that for a scheme to have the boundedness property its functional relationship

should be continuous and bounded from below by $\tilde{\phi}_f = \tilde{\phi}_c$ and from above by unity, should pass through the point (0,0) and (1,1) in the monotonic range $0 < \tilde{\phi}_c < 1$, and for $\tilde{\phi}_c < 0$ or $\tilde{\phi}_c > 1$ the functional relationship $f(\tilde{\phi}_c)$ should equal $\tilde{\phi}_c$. These conditions can be described graphically on a Normalized Variable Diagram (NVD) as shown in Figure 2.

This criterion can be imposed on any scheme, irrespective of the order of its interpolation profile. It will be used to ensure the boundedness of the seventh order scheme described below.

Bounded Seventh Order Scheme

The numerical dispersion problem that was plaguing the development of convective schemes based on high order interpolation profiles was resolved by imposing a flux limiter to the interpolated profile and the CBC criterion can actually be enforced on any interpolation profile to yield an equivalent but bounded version. In general, an interpolation profile can be constructed by fitting a polynomial (Figure 3(a)), as in the following equation, to a set of control volume nodes judiciously chosen:

$$\phi = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5 + \dots \quad (9)$$

In the above equation (Eq. (9)), ξ represents the local coordinate axis. The order of the polynomial can be chosen to yield the required scheme with its coefficients calculated by fitting the polynomial to a number of nodes in the computational domain. A seventh order scheme can thus be constructed by fitting a 6th order polynomial to the nodes at locations UUU, UU, U, C, D, DD, DDD (Figure 3(b)). The resulting polynomial is used to calculate the value of ϕ at the face of the control volume (ϕ_f). Then ϕ_c and ϕ_f are normalized to yield $\tilde{\phi}_f$ and $\tilde{\phi}_c$ respectively, and the CBC enforced in the event when it is not satisfied. The functional relationship for a seventh order scheme may be written for a regular grid as:

$$\phi_f = \frac{7}{1024}\phi_{DDD} - \frac{35}{512}\phi_{DD} + \frac{525}{1024}\phi_D + \frac{175}{256}\phi_C - \frac{175}{1024}\phi_U + \frac{21}{512}\phi_{UU} - \frac{5}{1024}\phi_{UUU} \quad (10)$$

and for an irregular grid as:

$$\begin{aligned}
\phi_f = & \frac{\prod_{NB \neq DDD} (\xi_f - \xi_{NB})}{\prod_{NB \neq DDD} (\xi_{DDD} - \xi_{NB})} \phi_{DDD} + \frac{\prod_{NB \neq DD} (\xi_f - \xi_{NB})}{\prod_{NB \neq DD} (\xi_{DD} - \xi_{NB})} \phi_{DD} + \frac{\prod_{NB \neq D} (\xi_f - \xi_{NB})}{\prod_{NB \neq D} (\xi_D - \xi_{NB})} \phi_D \\
& + \frac{\prod_{NB \neq C} (\xi_f - \xi_{NB})}{\prod_{NB \neq C} (\xi_C - \xi_{NB})} \phi_C + \frac{\prod_{NB \neq U} (\xi_f - \xi_{NB})}{\prod_{NB \neq U} (\xi_U - \xi_{NB})} \phi_U + \frac{\prod_{NB \neq UU} (\xi_f - \xi_{NB})}{\prod_{NB \neq UU} (\xi_{UU} - \xi_{NB})} \phi_{UU} \\
& + \frac{\prod_{NB \neq UUU} (\xi_f - \xi_{NB})}{\prod_{NB \neq UUU} (\xi_{UUU} - \xi_{NB})} \phi_{UUU}
\end{aligned} \tag{11}$$

As can be seen from Eqs. (10) and (11), evaluating ϕ_f becomes very expensive when using a seventh order scheme and the cost increases further with increasing the order of the interpolation profile. Therefore, the computational power needed to solve a three-dimensional problem using a VHR scheme will be prohibitive. However, if used only in important regions, the cost may be affordable.

Implementation of The Adaptive Schemes

When constructing a cost-effective highly accurate adaptive scheme, it is important to pay attention to its implementation, to make sure it can be easily coded in current programs, and that the solution of the discretized equations can be readily accomplished with high rate of convergence. For that purpose, two techniques are used in tandem, namely the Deferred Correction (DC) procedure of Rubin and Khosla [12] and the Normalized Weighting Factor (NWF) method developed by the authors [20].

Deferred-Correction procedure

Higher order schemes, such as QUICK, can be directly implemented in the discretization method presented earlier. However, for HR schemes the situation differs. Due to the composite nature of these schemes (i.e. they are based on a combination of polynomials), a direct substitution of their functional relationships cannot be performed and other alternatives should be sought.

One way to simplify the implementation of HR schemes in codes initially written for Hybrid-type schemes is through the use of a compacting procedure such as the DC method of Rubin and Khosla [12]. In this technique, Eq. (2) is rewritten as:

$$J_e^U + J_w^U + J_n^U + J_s^U = B + \underline{\left[C_e (\phi_e^U - \phi_e) + C_w (\phi_w^U - \phi_w) + C_n (\phi_n^U - \phi_n) + C_s (\phi_s^U - \phi_s) \right]} \tag{12}$$

where ϕ_f^U is the face value, J_f^U the total flux of ϕ_f both calculated using the first order upwind scheme, ϕ_f the cell face value calculated using the chosen higher order scheme, and the underlined

terms represent the extra source term due to the deferred correction. Substituting the value of the cell flux obtained from the functional relationship of the upwind and high-resolution scheme at hand, the deferred correction results in an equation similar in form to Eq. (7), but where the coefficient matrix is pentadiagonal (for two-dimensional situations) and always diagonally dominant since it is formed using the first order upwind scheme. The discretized equation, Eq. (7), becomes:

$$A_P \phi_P = \sum_{NB} A_{NB} \phi_{NB} + B_P + B_{dc} \quad (13)$$

where now the coefficients A_P and A_{NB} are obtained from a first order upwind discretization, $NB=(E,W,N,S)$, and B_{dc} is the extra deferred correction source term. This compacting procedure is simple to implement and effective when using higher-order schemes. However, as the difference between the cell face values calculated with the upwind scheme and that calculated with the HR scheme becomes larger, the convergence rate diminishes. This effect can be easily estimated on an NVD; the difference between the Upwind line and that of the chosen scheme is the normalized difference between the cell face values. The larger this difference is, the lower the convergence rate will be. The problem will be even magnified in three dimensional situations or when implementing VHR schemes and better alternatives are needed. The best available remedy, is the NWF method developed by Darwish and Moukalled [20] that is outlined next.

Normalized Weighting Factor method (NWF)

As mentioned above, the deferred correction method may result in large source terms that are explicitly calculated. It is this explicitness that slows down the convergence rate. The NWF technique resolves this weakness and eliminates the need for such a source term when implementing HR schemes by exploiting the virtues of the NVSF methodology.

Nearly all HR schemes can be written as a set of linear equations of the form:

$$\tilde{\phi}_f = m\tilde{\phi}_C + k \quad (14)$$

where m and k are constants within any interval of $\tilde{\phi}_f$, with the number of intervals depending on the HR scheme used. For example, by equating Eq. (18) to the NVSF form of the SMART scheme given by

$$\tilde{\phi}_f = \begin{cases} \frac{\tilde{\xi}_f(1 - 3\tilde{\xi}_C + 2\tilde{\xi}_f)}{\tilde{\xi}_C(1 - \tilde{\xi}_C)} \tilde{\phi}_C & 0 < \tilde{\phi}_C < \frac{\tilde{\xi}_C}{3} \\ \frac{\tilde{\xi}_f(1 - \tilde{\xi}_f)}{\tilde{\xi}_C(1 - \tilde{\xi}_C)} \tilde{\phi}_C + \frac{\tilde{\xi}_f(\tilde{\xi}_f - \tilde{\xi}_C)}{1 - \tilde{\xi}_C} & \frac{\tilde{\xi}_C}{3} \leq \tilde{\phi}_C < \frac{\tilde{\xi}_C}{\tilde{\xi}_f} (1 + \tilde{\xi}_f - \tilde{\xi}_C) \\ 1 & \frac{\tilde{\xi}_C}{\tilde{\xi}_f} (1 + \tilde{\xi}_f - \tilde{\xi}_C) \leq \tilde{\phi}_C < 1 \\ \tilde{\phi}_C & elsewhere \end{cases} \quad (15)$$

the values of m and k are found to be

$$[m, k] = \begin{cases} \left[\frac{\tilde{\xi}_f(1 - 3\tilde{\xi}_C + 2\tilde{\xi}_f)}{\tilde{\xi}_C(1 - \tilde{\xi}_C)}, 0 \right] & 0 < \tilde{\phi}_C < \frac{\tilde{\xi}_C}{3} \\ \left[\frac{\tilde{\xi}_f(1 - \tilde{\xi}_f)}{\tilde{\xi}_C(1 - \tilde{\xi}_C)}, \frac{\tilde{\xi}_f(\tilde{\xi}_f - \tilde{\xi}_C)}{1 - \tilde{\xi}_C} \right] & \frac{\tilde{\xi}_C}{3} \leq \tilde{\phi}_C < \frac{\tilde{\xi}_C}{\tilde{\xi}_f} (1 + \tilde{\xi}_f - \tilde{\xi}_C) \\ [0, 1] & \frac{\tilde{\xi}_C}{\tilde{\xi}_f} (1 + \tilde{\xi}_f - \tilde{\xi}_C) \leq \tilde{\phi}_C < 1 \\ [1, 0] & elsewhere \end{cases} \quad (16)$$

Furthermore, Eq. (18) could also be written as:

$$\frac{\phi_f - \phi_U}{\phi_D - \phi_U} = m \frac{\phi_C - \phi_U}{\phi_D - \phi_U} + k \quad (17)$$

which yields

$$\phi_f = m(\phi_C - \phi_U) + k(\phi_D - \phi_U) + \phi_U = m\phi_C + k\phi_D + (1 - m - k)\phi_U \quad (18)$$

where ϕ_U , ϕ_D and ϕ_C are the Upstream, Downstream and Central node values of ϕ near the control volume face at hand. Obviously these values depend on the flow direction. After some algebraic manipulations Eq. (2) is written in the following form

$$A_P \phi_P = \sum_{NB=E,W,N,S,EE,WW,NN,SS} A_{NB} \phi_{NB} + B_P \quad (19)$$

As can be seen, the resulting discretized equation (Eq. (19)) has a computational stencil involving five grid points in each coordinate direction which is solved here by applying iteratively the Penta-diagonal Matrix Algorithm (PDMA) in each coordinate direction. Moreover, it should be clear here that the NWF will not produce any additional source term if used to implement HR schemes. However, when higher order schemes are involved, the NWF method cannot be used directly.

Rather, the NWF can be combined with the deferred-correction technique to produce a deferred correction source term (B_{dc}) that is less important than it would be if the combination was between the HYBRID and the seventh-order scheme. Thus, the deferred correction will be between SMART and the BSEVENTH scheme which largely reduces its value and greatly enhances the convergence rate as revealed by the results shown later.

Normalized Adaptive Criterion

An important ingredient of any adaptive technique is the switching criterion, and how frequently it is applied. The switching criterion should be capable of defining the locations where the VHR scheme should be used. Since smooth regions in the flow field are characterized by lower gradient levels as compared to other regions a parameter that quantitatively measure the gradient level is a good choice. A widely used parameter in the literature [17] is the absolute value of the gradient across the cell face of the finite volume, given by

$$\text{GRAD}_f = \left| \frac{\phi_D - \phi_C}{\xi_D - \xi_C} \right| \quad (20)$$

The problems with using the GRAD parameter in this form can be illustrated through the following example (Figure 4). The GRAD parameter is evaluated over a uniformly discretized domain, for a one-dimensional problem involving two elliptic, two step, and two sine profiles (Figure 4(a)). Every two identical profiles have different maximum values, with one profile having a maximum that is twice the maximum of the other. From computations, it is known that for a purely convective flow problem results for the similar profiles are comparable in accuracy. However, using the GRAD parameter to decide on where to adapt, the profiles with higher maximum values will require the use of the wider stencil scheme since the GRAD value will be a factor of 2 greater than the GRAD of the profiles with the lower maximum values (Figure 4(b)). Therefore, the GRAD parameter cannot be used in an absolute manner to decide on where to adapt. This is to say that a universal value cannot be assigned to the GRAD parameter above which adaptation is to be performed. Since the grid is alike, the difference in the GRAD values are solely due to the variation in the levels of ϕ . Thus, to obtain the same GRAD values for two similar profiles, the ϕ field should be scaled or normalized. Another difficulty arises when calculating the GRAD parameter for the same profile using different grid networks (Figure 4(c)). Results displayed in Figure 4(d) shows the same trend noticed above. In this case, the difference in ξ affects the value of GRAD, with the

coarser grid (compare Figures 4(b) and 4(d)) showing a lower GRAD levels and thus not making use of the wider stencil scheme as much as the denser grid, which is contrary to the actual requirements. Hence the local coordinate should be normalized as well.

To overcome the above mentioned shortcomings, a new normalization procedure is adopted whereby the gradient over the interface is compared to the gradient obtained from the two next points as if the gradient of the interface is calculated using a coarser grid. In this case the normalized gradient parameter becomes:

$$\overline{\text{GRAD}}_f = \frac{\frac{|\phi_D - \phi_C|}{|\xi_D - \xi_C|}}{\frac{|\phi_{DD} - \phi_U|}{|\xi_{DD} - \xi_U|}} \quad (21)$$

The normalized gradient parameter $\left(\overline{\text{GRAD}}\right)$ is computed for the same profiles mentioned above using similar and different grid systems. Results displayed in Figure 5, reveal that the values of $\overline{\text{GRAD}}$ are the same in all cases indicating that a universal value can be set for $\overline{\text{GRAD}}$ beyond which adaptation can be performed. The only difficulty that arises is when the maximum value of the profile is very small, in which case, adaptation will be unnecessarily required. To overcome this problem, the GRAD value is used as a filtering parameter, i.e. if the numerator is less than a threshold value the normalized gradient $\left(\overline{\text{GRAD}}\right)$ is set to zero thus avoiding any artificially high value. This treatment is tested for a one-dimensional problem involving a sine, a step and an elliptic profile (Figure 6) in addition to some random oscillations between the profiles. The resulting GRAD values presented in Figure 6 show chaotic behaviour whereas the $\left(\overline{\text{GRAD}}\right)$ fields does not show such a behaviour indicating the effectiveness of this filtering procedure.

Strategies for Adaptive Schemes

The above criterion can be applied at each iteration in order to flag points where the SMART and BSEVENTH schemes should be used. However, it was found that such an approach is overly time consuming and can lead to oscillations in the solution field at locations near the limiting value of the criterion. In order to avoid these problems, another technique is adopted in this work whereby the switching occurs only after a near converged solution is obtained using the SMART scheme. Thus, the SMART scheme is invoked in solving the equations until the residual error drops to a set level $\text{ERROR}_{\text{SWITCH}}$, then the resulting solution field is employed to calculate the criterion for one time and to delimit regions where the higher order scheme should be applied. The solution

field is then driven to full convergence using the SMART scheme in the non- flagged regions and the BSEVENTH scheme in the flagged regions. This strategy was found to yield the lowest computational time because using the criterion before the residual of the field has been lowered enough is not useful since it would be based on values that are far from the converged solution field. Moreover, applying the adaptive criterion more frequently unnecessarily increases the computational cost without improving the accuracy.

Obviously, the values of $ERROR_{SWITCH}$ and $\overline{GRAD}$ affect the computational cost and accuracy of the final solution. Starting with $ERROR_{SWITCH}$, if it is set to a large value, the switching criterion will be applied to a field that is far from the converged one. On the other hand, since the final solution is to be obtained using the combination of SMART/BSEVENTH schemes, a very small value of $ERROR_{SWITCH}$ means that more time than needed has been taken to converge the solution using the SMART scheme. Similarly, a large value of $\overline{GRAD}$ would mean that some regions of high gradient levels may be missed without being flagged, while a small value may result in flagging a larger region than necessarily required. In order to decide on the proper values of $ERROR_{SWITCH}$ and $\overline{GRAD}$, experimentation were carried out for a number of problems. It was found that $ERROR_{SWITCH} = 50 ERROR_{MAX}$ and $\overline{GRAD} = 0.7$ result in the lowest computational cost and the most accurate solution, where ERROR is calculated from the following relation:

$$ERROR = \max_{i=1}^n \left[\left| a_p \phi_p - \left(\sum_{NB} a_{NB} \phi_{NB} + b_p + b_{dc} \right) \right| \right] \quad (22)$$

The above values were used in solving the test problems presented in this paper.

Test Problems

To check the performance of the new adaptive scheme, four purely convective problems are solved: convection of a step profile in an oblique velocity field, convection of a double-step profile in an oblique velocity field, convection of a sinusoidal profile in an oblique velocity field, and the Smith-Hutton problem. Results are obtained by covering the physical domains with uniform grids. Grid networks are generated using the transfinite interpolation technique [22]. In all tests, computational results are considered converged when the residual error (ERROR), as defined by Eq. (22), became smaller than a vanishing quantity.

Convection of a step profile in an oblique velocity field

Figure 7(a) shows the well known benchmark test problem consisting of a pure convection of a transverse step profile imposed at the inflow boundaries of a square computational domain. A 22x22 mesh system was used. The velocity vector was chosen to be at 45° to the horizontal and of magnitude 1 ($|V| = 1$).

The time required to solve the problem using the SMART, BSEVENTH, and the new adaptive scheme is displayed in Figure 7(b). The effect of using the NWF/DC in tandem is clearly shown by the difference in computational time between the NWF SMART and SMART, and the NWF BSEVENTH and BSEVENTH. Using the adaptive strategy yield a further decrease in computational time (about 57% lower than the NWF BSEVENTH) without affecting the accuracy of the solution. This is evident in Figure 7(c), where the profiles at $Y=0.925$ generated numerically by SMART, BSEVENTH, and the adaptive scheme are compared against the exact profile. As shown, the BSEVENTH and adaptive profiles fall on top of each others and are much more accurate than the SMART profile since they are closer to the exact profile.

The effectiveness of the adaptive criterion is clearly demonstrated by the flagged area displayed in Figure 7(d). As shown, only a small region including and surrounding the step is flagged.

Convection of a double-step profile in an oblique velocity field

A schematic of the physical situation is depicted in Figure 8(a). The flow field and mesh used are similar to those of the previous problem.

The double step profile is a more difficult variant of the oblique step profile and is shown to affect the performance of the NWF. The computational time of the NWF and DC implementation of the BSEVENTH scheme are closer than in the previous problem with the NWF implementation being about 20.5 % cheaper (Figure 8(b)). The adaptive implementation is clearly less expensive (about 58% lower than the NWF BSEVENTH). This large decrease in computational cost is accomplished at no degradation in the solution accuracy. As depicted in Figure 8(c), the adaptive and seventh profiles at $Y=0.775$ are much closer to the exact profile than SMART and fall on top of each others. The validity of the adaptive criterion is demonstrated in Figure 8(d). As shown, only regions around the steps are flagged.

Convection of a sinusoidal profile in an oblique velocity field

A sinusoidal profile was also used for the same geometric situation. This third problem, illustrated in Figure 9(a), was used in order to test the performance of the different schemes for a profile with an initial steep gradient that gradually decreases and then increases. The sinusoidal profile is generated using the following equation:

$$\phi = \sin\left(\frac{i\pi}{L_i}\right) \quad 1 < i < 10 \quad \text{and} \quad L_i = 5 \quad (23)$$

The same mesh as for the previous two test problems was used. As before, the computational time needed to solve the problem using the various schemes are shown in Figure 9(b). The NWF implementation of the BSEVENTH scheme is only 6% less expensive than its DC implementation. The adaptive solution is, however, 47.67% cheaper than the NWF-BSEVENTH solution. This decrease in computational cost is again achieved without any loss in the solution accuracy as shown by the profiles at $Y=0.925$ displayed in Figure 9(c). Even though the NWF-SMART solution is the least expensive, its accuracy is far below the adaptive NWF-SMART/BSEVENTH solution. Once more, the effectiveness of the adaptive criterion is clearly illustrated by the flagged regions displayed in Figure 9(d).

Smith-Hutton problem

In the fourth test problem shown in Figure 10(a), a step discontinuity at $x = -0.5$ is convected clockwise from the inlet plane ($x < 0, y = 0$) to the outlet plane ($x > 0, y = 0$) by a rotational velocity field given by:

$$\begin{aligned} u &= 2y(1-x^2) \\ v &= -2x(1-y^2) \end{aligned} \quad (23)$$

This test was devised for evaluating a number of numerical models of convection at the third meeting of the International Association for Hydraulic Research Working Group on Refined Modeling of Flow [23]. The boundary conditions for the problem are:

$$\phi = \begin{cases} 0 & \text{for } -0.5 < x < 0 & y=0 \\ 2 & \text{for } -1 < x < -0.5 & y=0 \\ 2 & \text{for } -1 < x < 1 & y=1 \\ 2 & \text{for } x=-1 & 0 < y < 1 \\ 2 & \text{for } x=1 & 0 < y < 1 \end{cases} \quad (24)$$

In this test a mesh composed of 42×22 grid points was used. Comparison of the time needed to solve the problem by the various schemes is depicted in Figure 10(b). The NWF implementation of the BSEVENTH scheme is about 58% less expensive than its DC implementation. The adaptive solution reduces the cost further by 39% of the time taken by the NWF-BSEVENTH solution without affecting the accuracy as shown by the profiles at $Y=0.025$ displayed in Figure 10(c). The ability of the adaptive criterion to capture locations where the very high order scheme should be used is evident by the flagged regions displayed in Figure 10(d).

Closing Remarks

In this paper, a new Adaptive Very High Resolution (AVHR) scheme was developed and tested. The new adaptive scheme is a combination of the third order SMART scheme and a bounded version of the seventh order scheme that was denoted by BSEVENTH. A new adaptive criterion for switching between the SMART and BSEVENTH schemes was devised. The performance of the AVHR scheme was checked in terms of computational time and accuracy by solving four purely convective flow problems. While preserving the accuracy of the BSEVENTH scheme, the AVHR scheme produced an average saving in computational time of about 50%.

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Figure Captions

Figure 1: (a) Control Volume; (b) Normalized Variables; (c) Control-volume nodes notation.

Figure 2: CBC criterion.

Figure 3: Interpolation profiles.

Figure 4: Problems with using the GRAD as criterion.

Figure 5: Criterion test.

Figure 6: Using the threshold value to counter oscillation effects.

Figure 7: Pure convection of a step profile.

Figure 8: Pure convection of a double step profile.

Figure 9: Pure convection of a sinusoidal profile.

Figure 10: Pure convection in a rotational field.

Figures

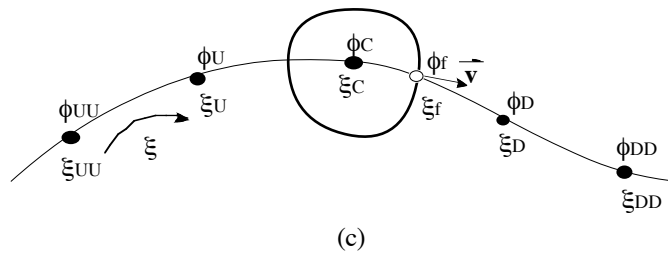
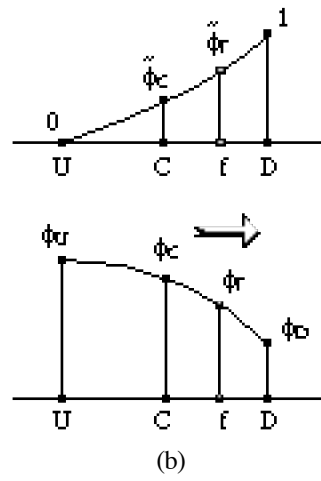
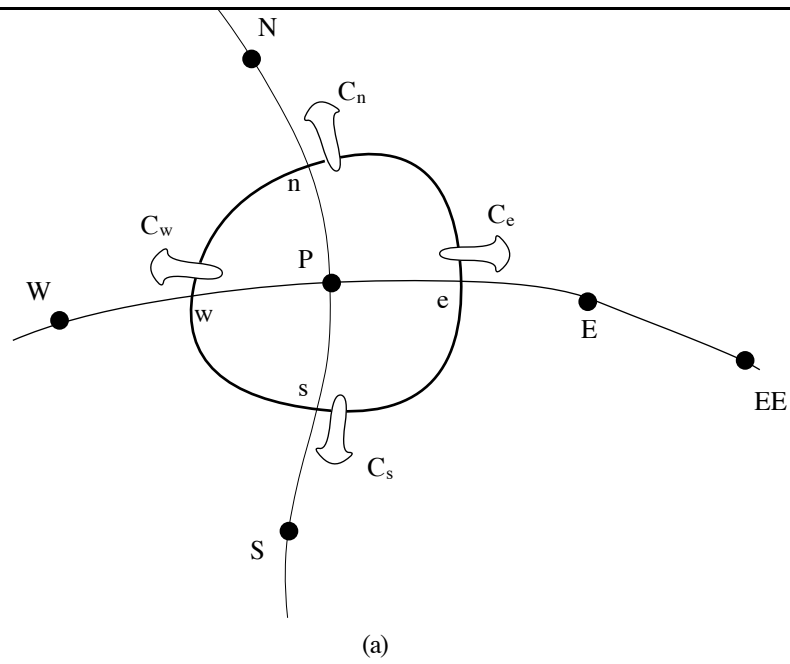


Figure 1: (a) Control Volume; (b) Normalized Variables; (c) Control-volume nodes notation.

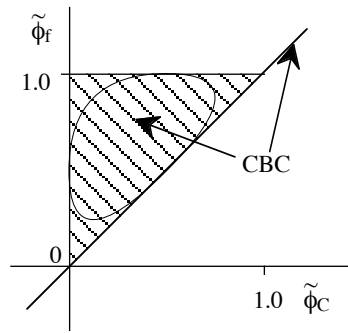


Figure 2: CBC criterion.

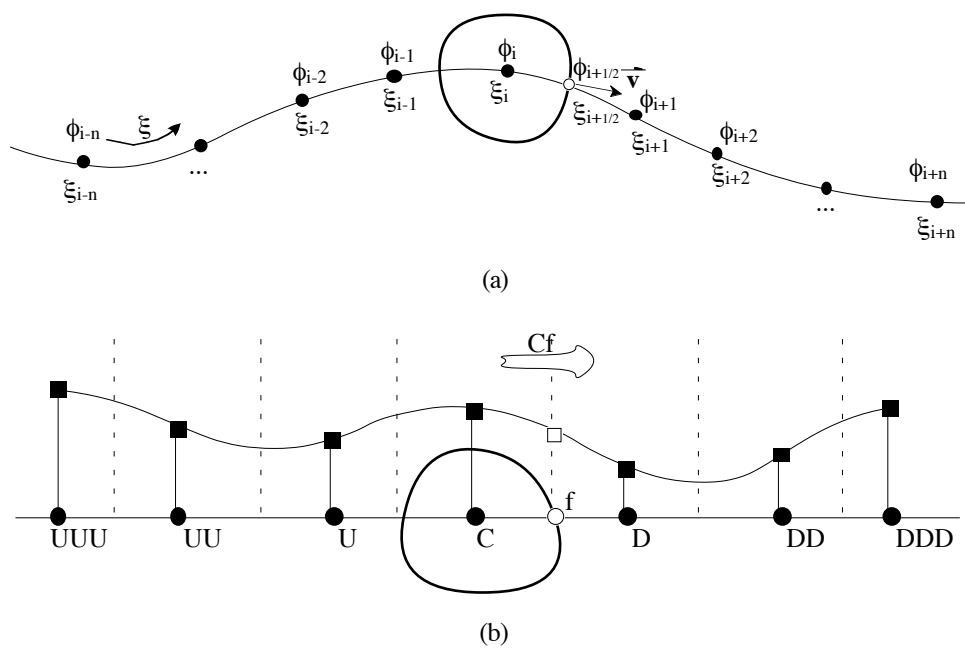


Figure 3: Interpolation profiles.

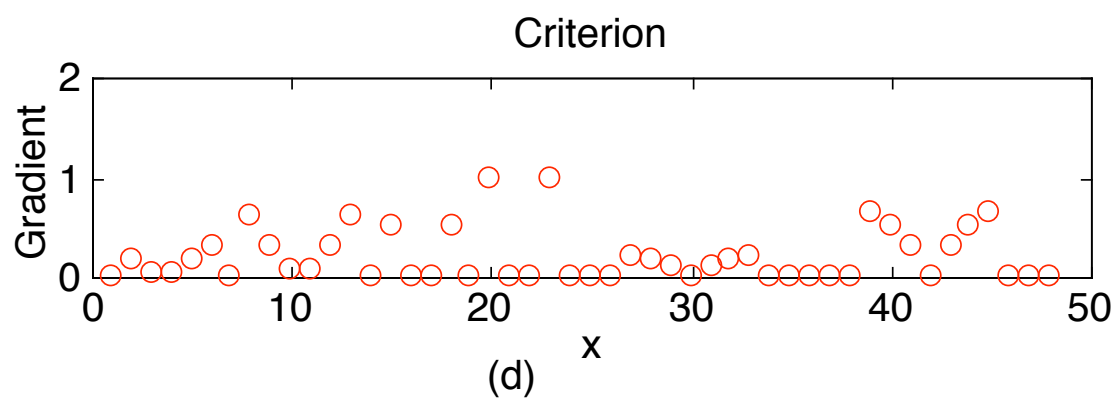
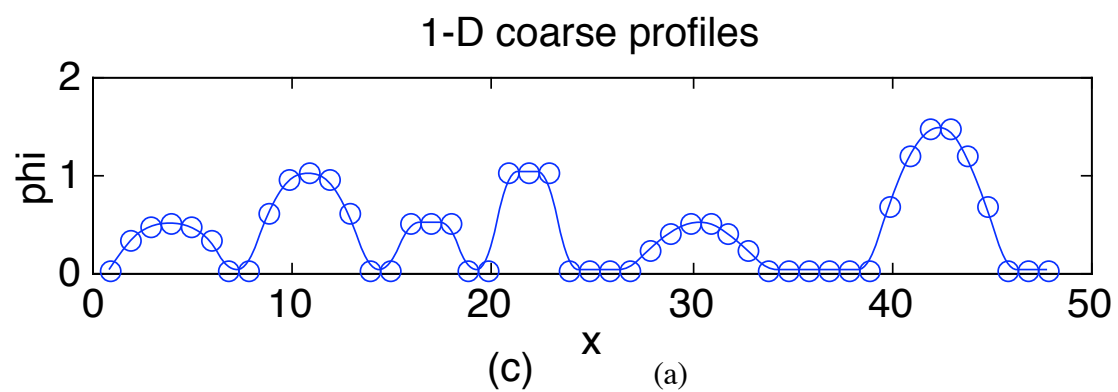
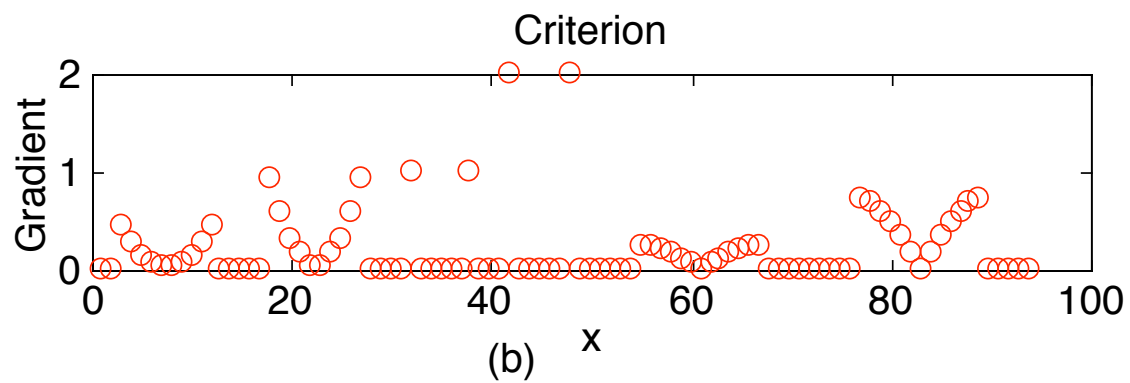
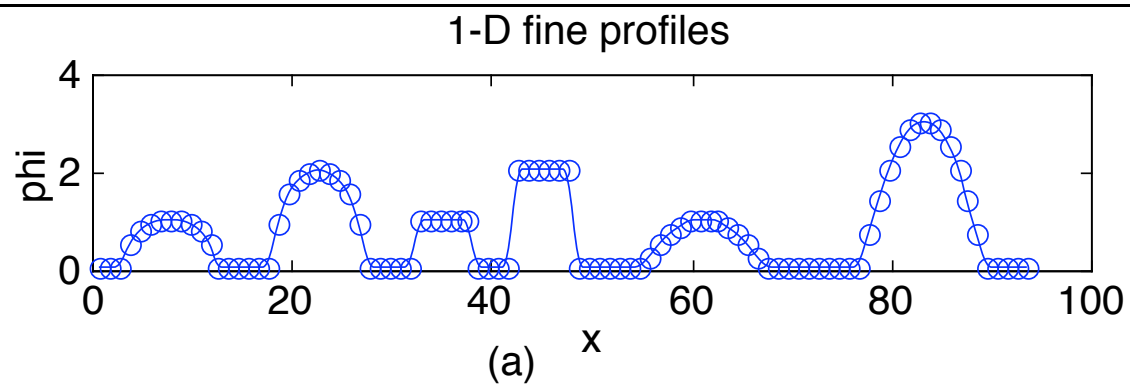


Figure 4: Problems with using the GRAD as criterion.

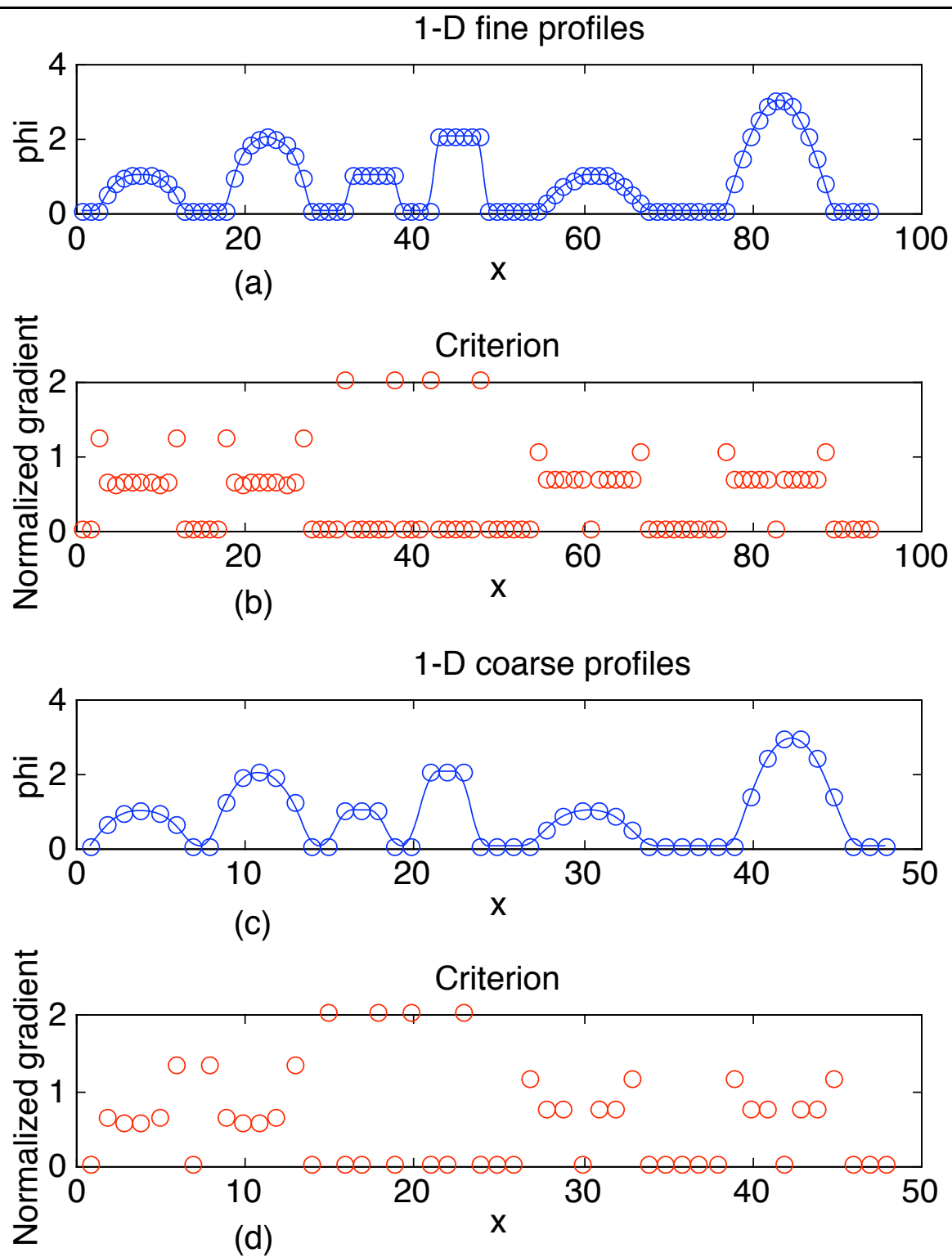


Figure 5: Criterion test.

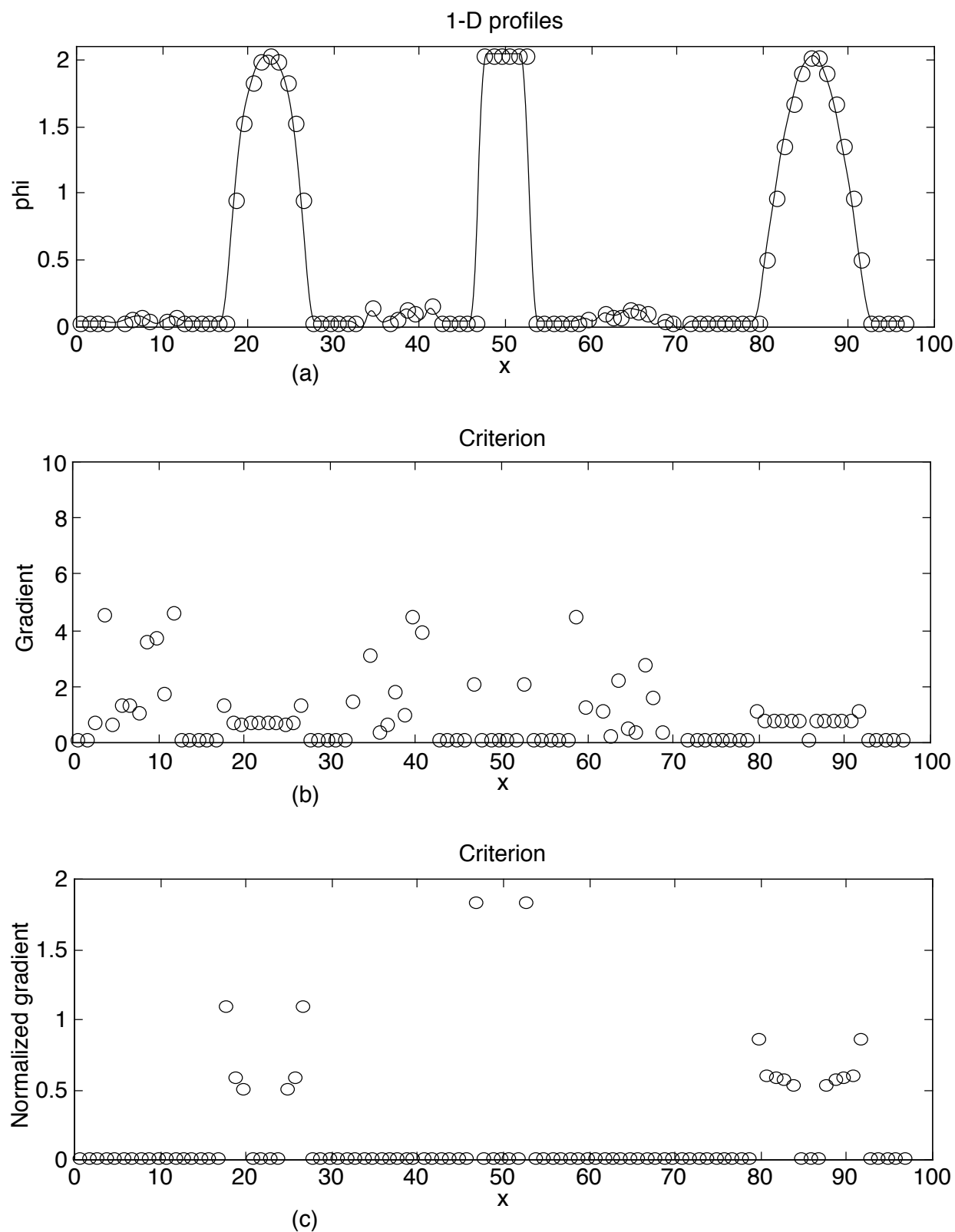


Figure 6: Using the threshold value to counter oscillation effects.

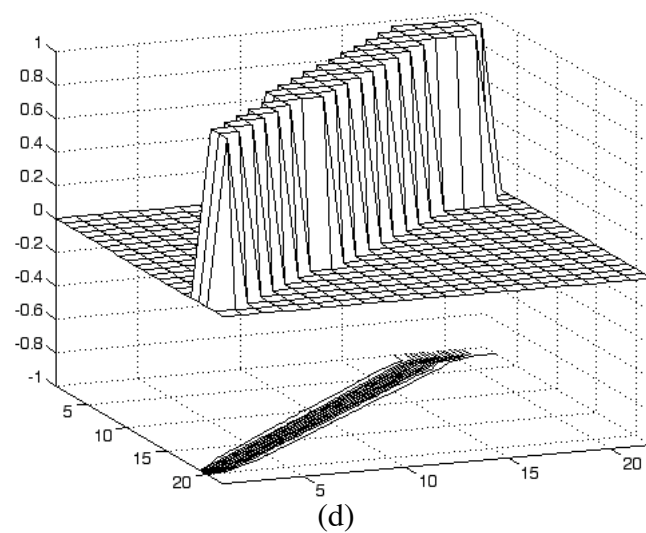
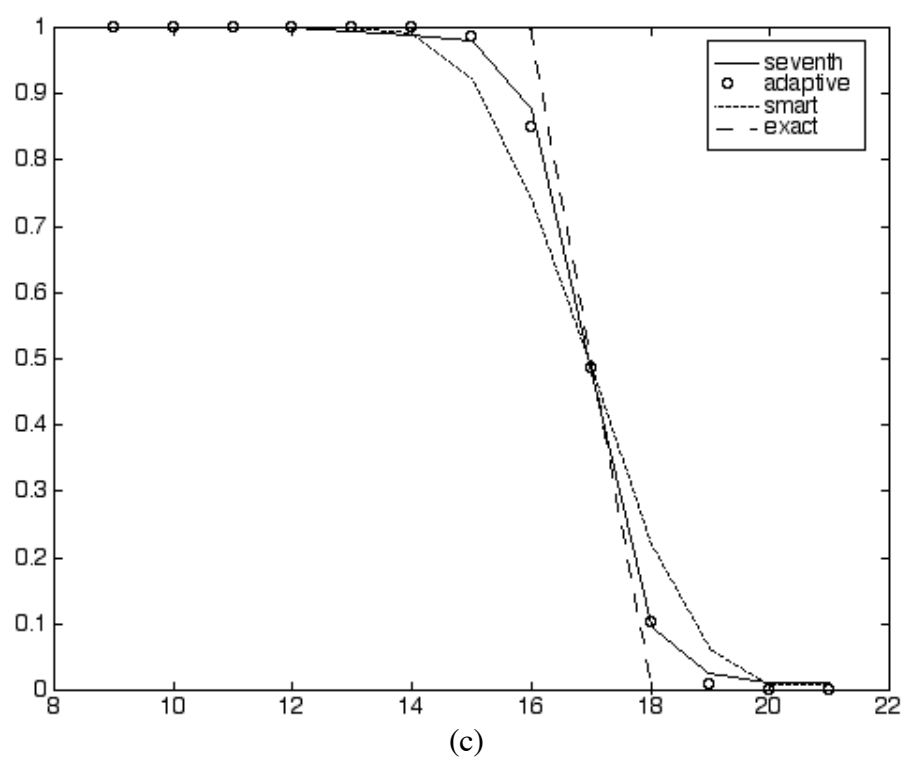
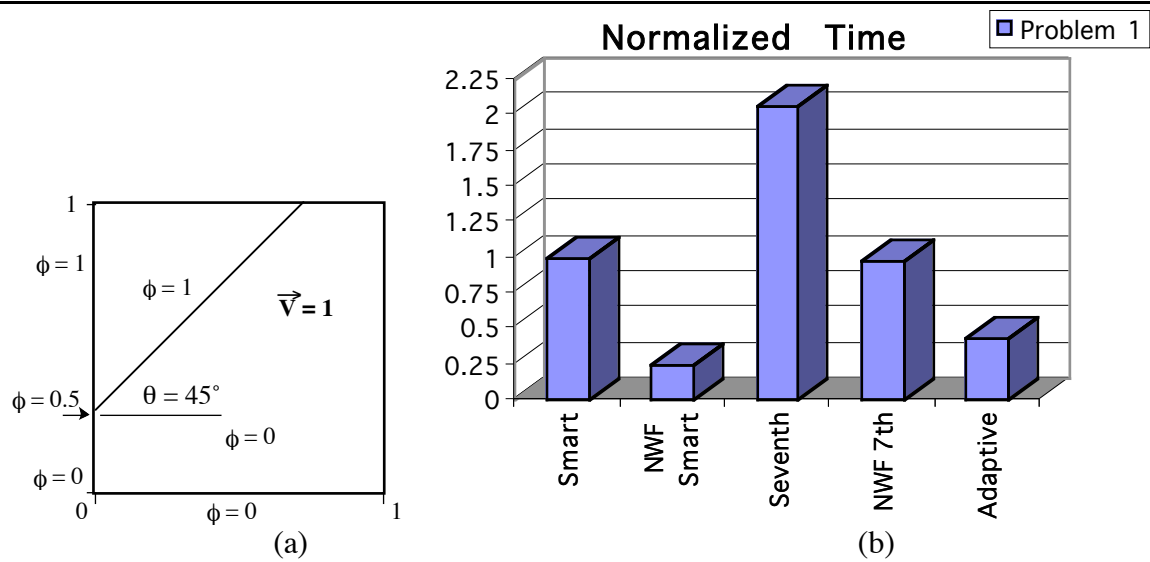


Figure 7: Pure convection of a step profile.

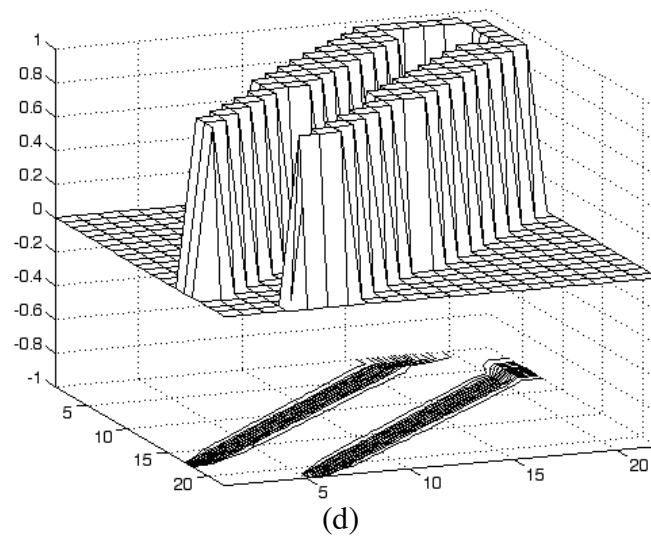
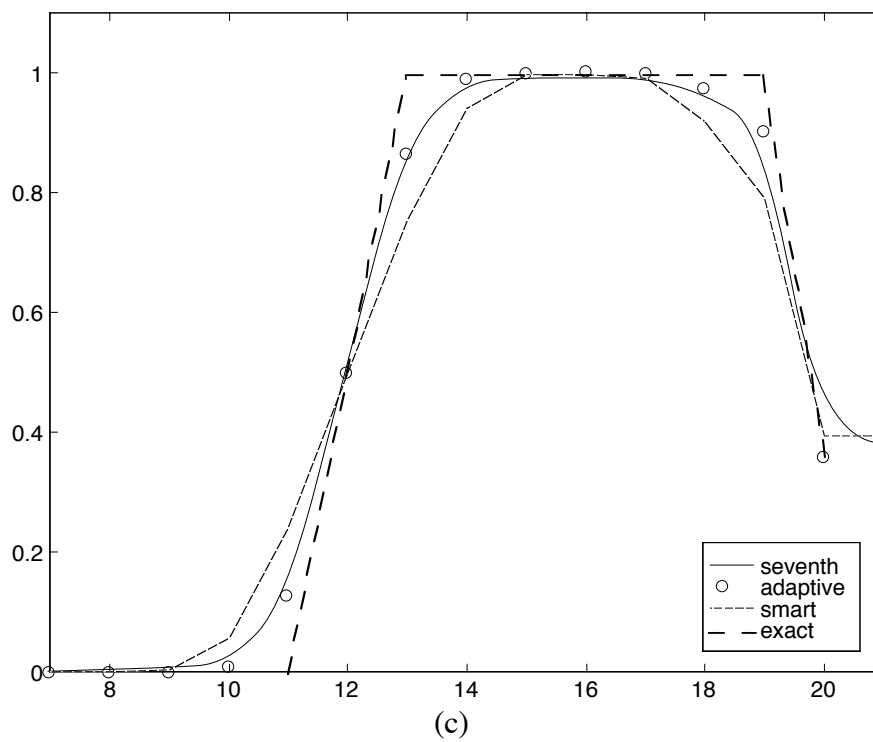
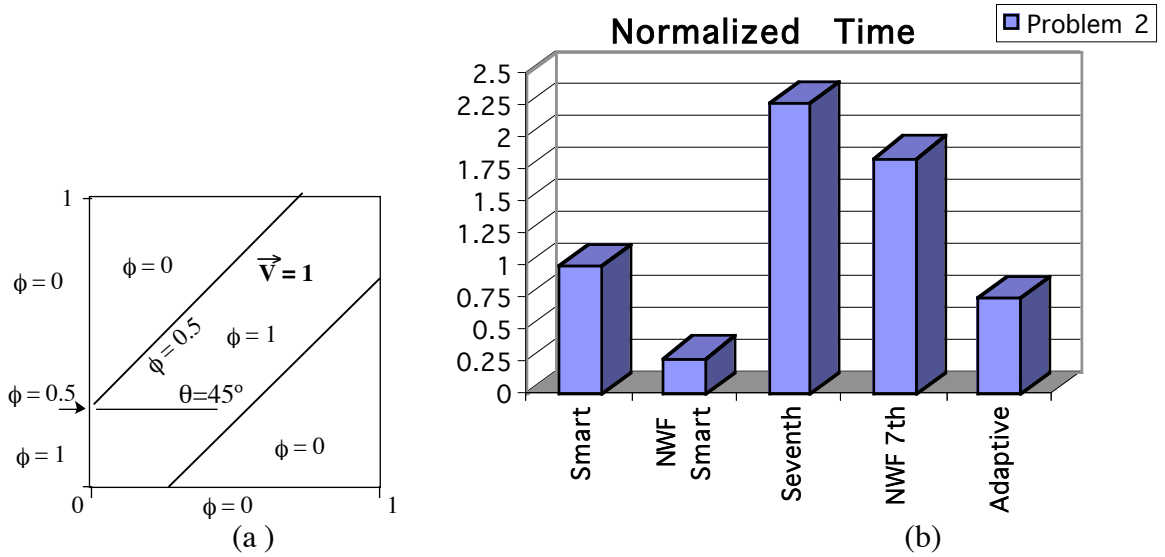


Figure 8: Pure convection of a double step profile.

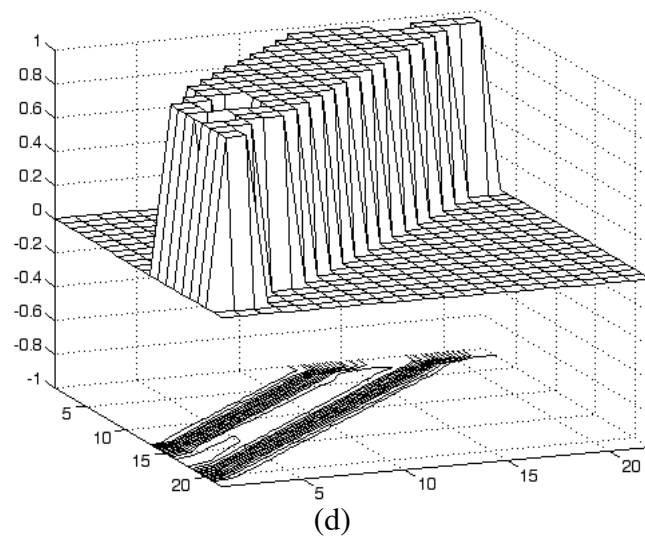
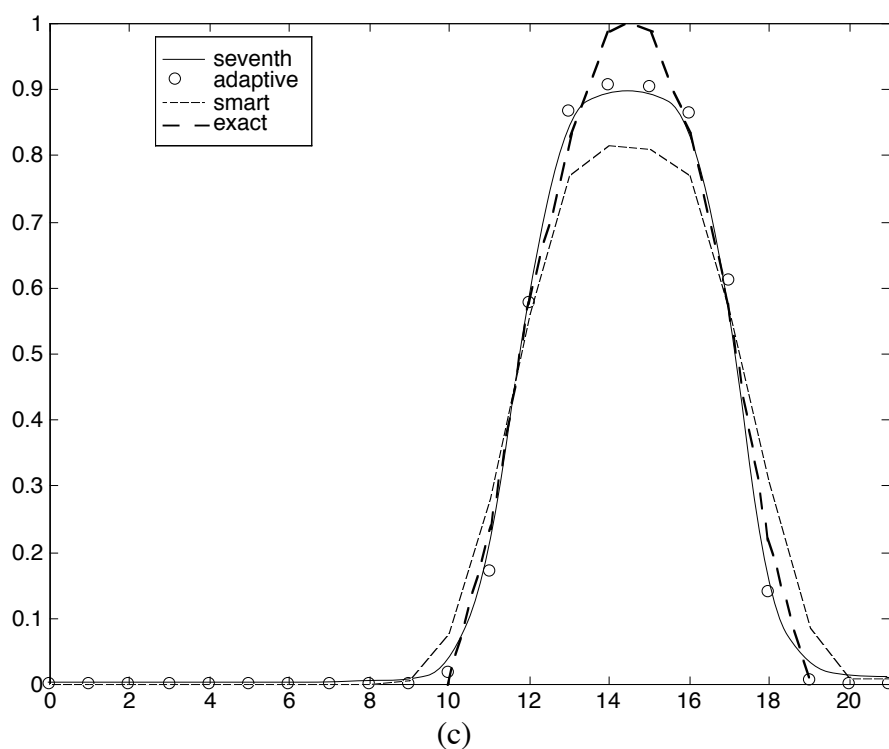
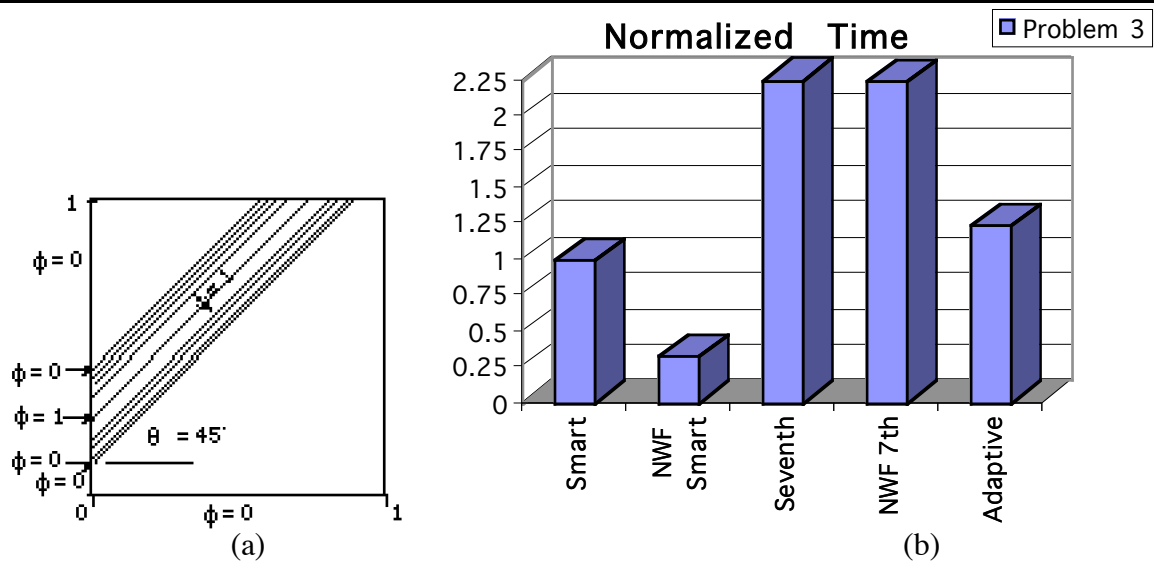


Figure 9: Pure convection of a sinusoidal profile.

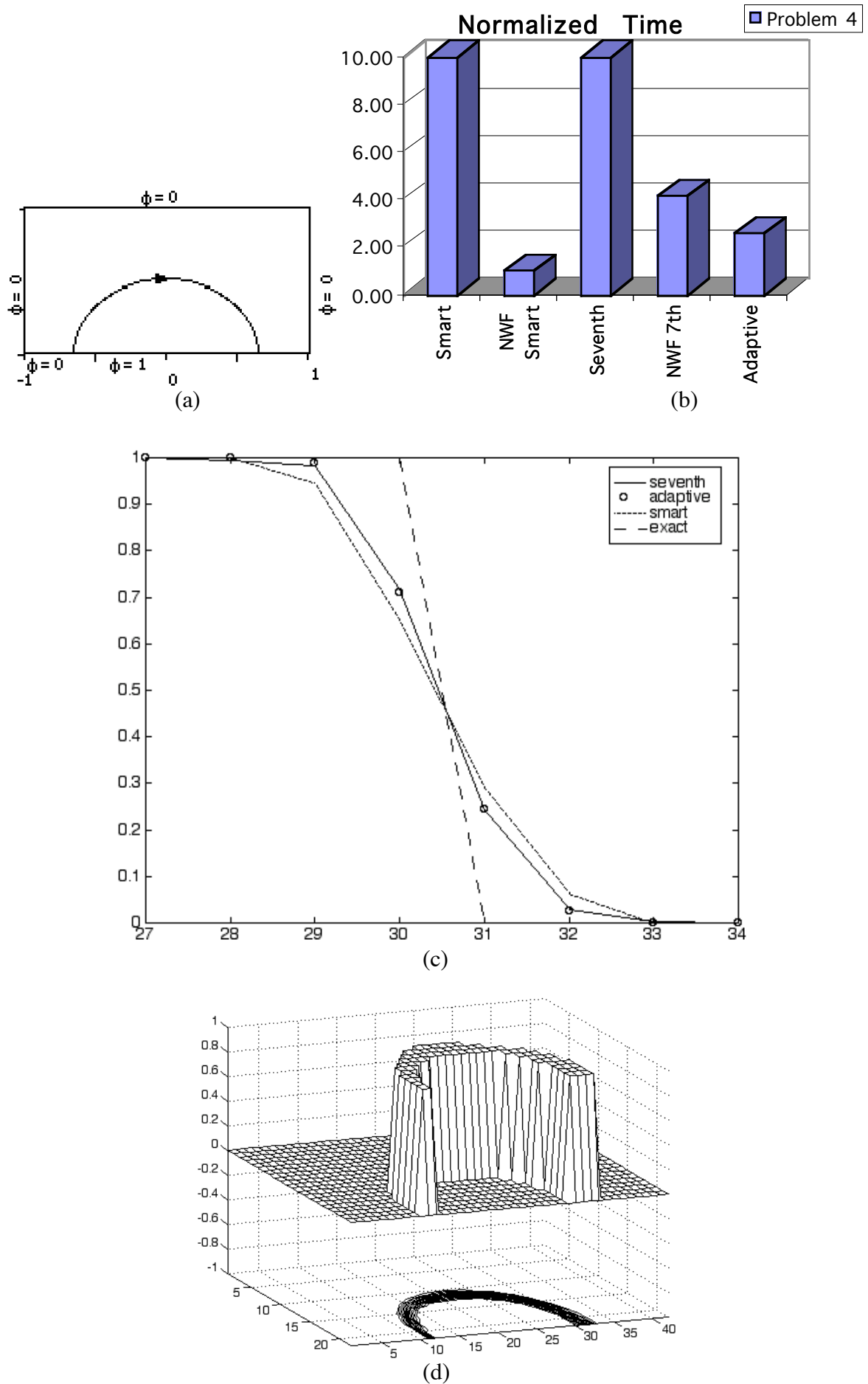


Figure 10: Pure convection in a rotational field.

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