INTERNATIONAL JOURNAL OF ENERGY RESEARCH, VOL. 20, 203-214 (1996)

A UNIVERSAL MODEL FOR STUDYING THE PERFORMANCE OF CARNOT-LIKE ENGINES AT MAXIMUM POWER CONDITIONS

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SUMMARY

A universal model for studying the performance of endoreversible heat engines with heat leak is presented. By exploiting finite-time thermodynamics, the new model allows detailed analyses of Carnot-like engines under combined modes of heat supply and/or release. Many established laws and major conclusions derived in several references are shown to represent very special cases of the new formulation. Furthermore, as a special case of the general model, the performance of endoreversible heat engines under combined conduction, convection, and radiation heat transfer modes is studied and generated results are displayed graphically.

KEY WORDS: endoreversible thermodynamics; finite-time thermodynamics; endoreversible engines with heat leak; carnot-like engines with heat leak; energy conversion systems

INTRODUCTION

Several studies have reported on the efficiency of Carnot-like heat engines at maximum power conditions. The common feature of these investigations deals with the additional limitation on efficiency imposed by the rate at which heat can be exchanged between the working material and the heat reservoirs. This additional resistance to heat flow, was first accounted for by Curzon and Ahlborn (1975) who employed Newton's law to model the heat flux across the walls of the hot and cold reservoirs. Chen and Yan (1989) generalized the work reported in Curzon and Ahlborn (1975) by assuming the rate of heat flowing through the walls of the reservoirs to be ruled by an equation of the form:

$$Q = \alpha(T_1^n - T_2^n) \tag{1}$$

where n is a nonzero integer. The use of equal powers to describe the rates of heat in and out has limited the applicability of results to those situations where similar heat transfer modes govern the hot and cold sides of the engine. DeVos (1985) simplified the analysis presented in Curzon and Ahlborn (1975) and Chen and Yan (1989) developing an easier model. Even though this model was introduced in general terms, only specific cases of limited usefulness were analysed. A number of workers (Gordon, 1991; Gordon and Zarmi, 1989; DeVos, 1991; Nuwayhid and Moukalled, 1994) have also applied finite-time thermodynamics to predicting phenomena of practical interest. Gordon (1991) applied finite-time thermodynamics to analyse the thermoelectric generator. Gordon and Zarmi (1989) modelled the Earth and its envelope using a Carnot engine with its heat input being solar radiation and its work output representing the wind generated. From these basic considerations, they derived a theoretical upper bound for the annual average wind energy on Earth. DeVos (1991) developed a simplified version of Gordon and Zarmi's model and applied it to studying the conversion efficiency of solar energy into wind energy. Nuwayhid and Moukalled (1994) added a heat leak term into the model of DeVos (1991) and studied the effect of a planet thermal conductance on conversion efficiency of solar energy into wind energy. The theoretical upper bound on conversion efficiency reported in DeVos (1991) was shown to be

CCC 0363-907X/96/030203-12

Received 1 July 1994
Revised 25 July 1994

well above the actual values predicted by the modified model. Nulton *et al.* (1993) and Pathria *et al.* (1993) described a set of feasible operations of a finite-time heat engine subject only to thermal losses in terms of an inequality similar to the second law of thermodynamics and applied it to Carnot-like refrigerators and heat pumps. Recently, Moukalled *et al.* (1995) generalized the Curzon–Ahlborn concept by adding a heat leak term into the DeVos model (1985). Their work was comprehensive in the sense that it allowed the use of different heat transfer power laws for the various heat transfer processes involved. The model, however, was not capable of accommodating combined modes of heat transfer.

It is the intention of this work to generalize Moukalled *et al.*'s model (1995) and extend it into situations where the exchange of heat between the engine, heat reservoirs, and the surroundings occurs via combined modes of heat transfer. As will be seen, this generalization results in very complicated equations which, in general, have to be tackled numerically. Moreover, a variety of well-established formulae, such as the Curzon–Ahlborn efficiency, the Castans efficiency etc., are shown to represent very special cases of the general results presented in this work.

THE GENERALIZED HEAT-LEAK MODEL

A schematic of the endoreversible engine under consideration is depicted in Figure 1. As shown, heat transfer is assumed to be directly proportional neither to temperature nor to temperature raised to the fourth power as is conventionally done. Rather, multiple powers of temperature (or combined heat transfer modes) are assumed to be involved in the heat transfer process. The implication of such a spectrum of heat transfer modes is not the subject of the paper. As such, the focus will be on the new model and general solution methodologoy presented.

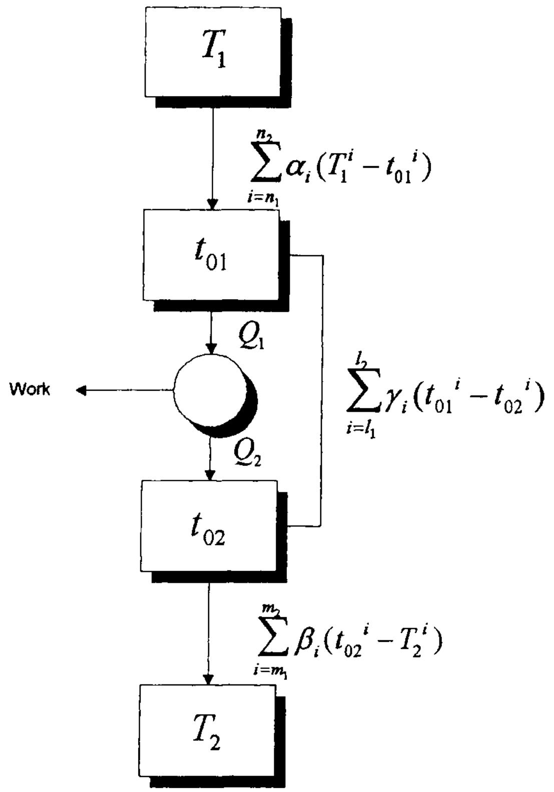

Figure 1. Schematic model of endoreversible engine with heat leak operating between hot reservoir at T_1 and cold reservoir at T_2

Irreversible heat transfer takes place between the heat source and the hot reservoir of the engine (Figure 1), while heat transfer from the hot reservoir to the engine is considered to occur reversibly. An analogous situation exists between the engine and the heat sink. The friction loss within the engine and the heat loss between the engine and the surroundings are reversibly modelled via the heat-leak term. With such a model, the heat transfer from the hot reservoir into the engine Q_1 is given by a heat balance equation as

$$Q_1 = \sum_{i=n_1}^{n_2} \alpha_i (T_1^i - t_{01}^i) - \sum_{i=l_1}^{l_2} \gamma_i (t_{01}^i - t_{02}^1) \tag{2}$$

while the heat transfer from the engine to the cold reservoir is

$$Q_2 = \sum_{i=m_1}^{m_2} \beta_i (t_{02}^i - T_2^i) - \sum_{i=l_1}^{l_2} \gamma_i (t_{01}^i - t_{02}^i) \tag{3}$$

where α_i, β_i, and γ_i are the appropriate coefficients of heat transfer and the superscript i (which may assume any value between n_1 and n_2, m_1 and m_2, of l_1 and l_2) designates the model power of the heat transfer law. Endoreversibility requires that,

$$\frac{Q_1}{t_{01}} = \frac{Q_2}{t_{02}} \tag{4}$$

with the Carnot efficiency given by

$$\eta = 1 - \frac{t_{02}}{t_{01}} \tag{5}$$

The work can therefore be found from the following relation:

$$W = \sum_{i=n_1}^{n_2} \alpha_i (T_1^i - t_{01}^i) - \sum_{i=m_1}^{m_2} \beta_i (t_{02}^i - T_2^i) \tag{6}$$

Applying the reversibility condition by inserting Q_1 and Q_2 into equation (4), the following equation for t_{01} in terms of η, T_1, T_2, α_i, β_i, and γ_i is obtained:

$$\sum_{i=n_1}^{n_2} \alpha_i (1-\eta) t_{01}^i + \sum_{i=m_1}^{m_2} \beta_i (1-\eta)^i t_{01}^i - \sum_{i=l_1}^{l_2} \gamma_i \left[\eta - \eta(1-\eta)^i\right] t_{01}^i - \sum_{i=m_1}^{m_2} \alpha_i (1-\eta) T_1^i - \sum_{i=m_1}^{m_2} \beta_i T_2^i = 0 \tag{7}$$

The above equations may be written in dimensionless forms by defining the following dimensionless quantities:

$$V_i = \frac{\alpha_i T_1^i}{\sum_{i=n_1}^{n_2} \alpha_i T_1^i}, \quad R_j = \frac{\beta_i T_1^i}{\sum_{i=n_1}^{n_2} \alpha_i T_1^i}, \quad S_i = \frac{\gamma_i T_1^i}{\sum_{i=n_1}^{n_2} \alpha_i T_i^1}, \quad \tau = \frac{T_2}{T_1}, \quad \text{and} \quad t_1 = \frac{t_{01}}{T_1} \tag{8}$$

In matrix notation V_i, R_i, and S_i may be written as:

$$V_i = [V_{n1}\ V_{n1+1}\ V_{n1+2} \ldots V_{n2-1}\ V_{n2}] \tag{9}$$

$$R_i = [R_{n1}\ R_{n1+1}\ R_{n1+2} \ldots R_{n2-1}\ R_{n2}] \tag{10}$$

$$S_i = [S_{n1}\ S_{n1+1}\ S_{n1+2} \ldots S_{n2-1}\ S_{n2}] \tag{11}$$

From the above definition of V_i it is easily seen that

$$\sum_{i=n_1}^{n_2} V_i = V_{n1} + V_{n1+1} + V_{n1+2} + \ldots + V_{n2-1} + V_{n2} = 1 \tag{12}$$

Using the above dimensionless quantities along with the Carnot efficiency (equation (5)), the efficiency and work equations (equations (7) and (6) are transformed respectively to

$$\sum_{i=n_1}^{n_2}(1-\eta)V_i t_1^i + \sum_{i=m_1}^{m_2} R_i(1-\eta)^i t_1^i - \sum_{i=l_1}^{l_2} S_i\left[\eta-\eta(1-\eta)^i\right]t_1^i - (1-\eta) - \sum_{i=m_1}^{m_2} R_i\tau^i = 0 \quad (13)$$

and

$$\frac{W}{\sum_{i=n_1}^{n_2}\alpha_i T_1^i} = 1 + \sum_{i=m_1}^{m_2} R_i\tau^i - \sum_{i=n_1}^{n_2} V_i t_1^i - \sum_{i=m_1}^{m_2} R_i(1-\eta)^i t_1^i \quad (14)$$

Since i may vary between $-\infty$ and $+\infty$, analytical solutions may be obtained for some specific situations. In general however, solutions should be obtained numerically.

SOLUTION METHODOLOGY

The general equations 913) and (14) are used in this section to obtain the efficiency of the endoreversible heat engine at maximum power. For this purpose, the derivative of the normalized power equation (equation (14)) with respect to the efficiency is set to zero. This results in the following general relation:

$$\frac{dt_1}{d\eta} = \frac{\sum_{i=m_1}^{m_2} iR_i(1-\eta)^{i-1}t_1^i}{\sum_{i=n_1}^{n_2} iV_i t_1^{i-1} + \sum_{i=m_1}^{m_2} iR_i(1-\eta)^i t_1^{i-1}} \quad (15)$$

A second equation for the derivative of t_1 with respect to η may be obtained from the reversibility equation, equation (13), and is given by:

$$\frac{dt_1}{d\eta} = \frac{\sum_{i=n_1}^{n_2} V_i t_1^i + \sum_{i=m_1}^{m_2} iR_i(1-\eta)^{i-1}t_1^i + \sum_{i=l_1}^{l_2} S_i\left[1+(1-\eta)^{i-1}(\eta+i\eta-1)\right]t_1^i - 1}{\sum_{i=n_1}^{n_2} iV_i(1-\eta)t_1^{i-1} + \sum_{i=m_1}^{m_2} iR_i(1-\eta)^i t_1^{i-1} - \sum_{i=l_1}^{l_2} iS_i\left[\eta-\eta(1-\eta)^i\right]t_1^{i-1}} \quad (16)$$

Equating equations (15) and (16), a relation for the efficiency at maximum power (η_m) is obtained and its final form is written as:

$$\frac{\sum_{i=m_1}^{m_2} iR_i(1-\eta_m)^{i-1}t_1^i}{\sum_{i=n_1}^{n_2} iV_i t_1^{i-1} + \sum_{i=m_1}^{m_2} iR_i(1-\eta_m)^i t_1^{i-1}}$$

$$= \frac{\sum_{i=n_1}^{n_2} V_i t_1^i + \sum_{i=m_1}^{m_2} iR_i(1-\eta_m)^{i-1}t_1^i + \sum_{i=l_1}^{l_2} S_i\left[1+(1-\eta_m)^{i-1}(\eta_m+i\eta_m-1)\right]t_1^i - 1}{\sum_{i=n_1}^{n_2} iV_i(1-\eta_m)t_1^{i-1} + \sum_{i=m_1}^{m_2} iR_i(1-\eta_m)^i t_1^{i-1} - \sum_{i=l_1}^{l_2} iS_i\left[\eta_m-\eta_m(1-\eta_m)^i\right]t_1^{i-1}} \quad (17)$$

As the same time, the efficiency should satisfy the reversibility equation (equation (13)). This results in a highly nonlinear system of two equations in the two unknowns t_1 and η_m. Therefore, the problem is mathematically well defined and in general, the solution may be carried out numerically. However, for some special cases of practical interest, the above system may be reduced to a single equation in η_m for which the solution may be found numerically. Furthermore, earlier results reported in Curzon and Ahlborn (1975), Chen and Yan (1989), DeVoc (1985) and Moukalled *et al.* (1995) are shown to represent very special cases of this general formulation.

Combined conduction, convection, and radiation heat transfer modes

Since it is not feasible to perform a full parametric investigation, solutions are obtained for the situation where exchange of heat between the engine, heat reservoirs, and surroundings occurs via a combination of the well known conduction, convection, and radiation heat transfer modes. Under such conditions, the only nonzero powers are 1 and 4. As such, V_i, R_i, and S_i assume the following forms:

$$V_i = [V_1\ 0\ 0\ V_4], \qquad V_1 + V_4 = 1 \tag{18}$$

$$R_i = [R_1\ 0\ 0\ R_4] \tag{19}$$

$$S_i = [S_1\ 0\ 0\ S_4] \tag{20}$$

Furthermore, the dimensionless work equation (equation (14)) reduces to:

$$\frac{W}{\alpha_1 T_1 + \alpha_4 T_1^4} = 1 + R_1\tau + R_4\tau^4 - V_1 t_1 - V_4 t_1^4 - R_1(1-\eta)t_1 - R_4(1-\eta)^4 t_1^4 \tag{21}$$

and by setting the derivative of the work given by the above equation to zero, an equation for the derivative of t_1 with respect to η, similar to that given by equation (15), is obtained as follows:

$$\frac{\mathrm{d}t_1}{\mathrm{d}\eta} = \frac{R_1 t_1 + 4(1-\eta)^3 R_4 t_1^4}{(1-\eta)R_1 + 4(1-\eta)^4 R_4 t_1^3 + V_1 + 4t_1^3 V_4} \tag{22}$$

Similarly, by employing the relevant values for n_1, n_2, m_1, m_2, l_1, l_2, V_i, R_i, and S_i equation (13), (16), and (17) are transformed respectively to

$$a_1 t_1^4 + b_1 t_1 + c_1 = 0 \tag{23a}$$

where

$$a_1 = \eta^5 S_4 + \eta^4(R_4 - 4S_4) + \eta^3(-4R_4 + 6S_4) + \eta^2(6R_4 - 4S_4) - \eta(4R_4 + V_4) + R_4 + V_4 \tag{23b}$$

$$b_1 = -\eta^2 S_1 - \eta(R_1 + V_1) + R_1 + V_1 \tag{23c}$$

$$c_1 = \eta - R_4\tau^4 - R_1\tau - 1 \tag{23d}$$

and

$$\frac{\mathrm{d}t_1}{\mathrm{d}\eta} = \frac{\begin{array}{l} -1 + t_1(R_1 + 2\eta S_1 + V_1) + t_1^4 \\ \quad [4R_4 + \eta^2(12R_4 - 18S_4) - 5\eta^4 S_4 + \eta(-12R_4 + 8S_4) + \eta^3(-4R_4 + 16S_4) + V_4] \end{array}}{D} \tag{24a}$$

where

$$\begin{aligned} D = {} & R_1 - \eta^2 S_1 - \eta(R_1 + V_1) + V_1 \\ & + t_1^3[4R_4 + \eta^4(4R_4 - 16S_4) + \eta^2(24R_4 - 16S_4) + 4\eta^5 S_4 \\ & + \eta^3(-16R_4 + 24S_4) - \eta(16R_4 + 4V_4) + 4V_4] \end{aligned} \tag{24b}$$

and

$$a_2 t_1^7 + b_2 t_1^4 + c_2 t_1^3 + d_2 t_1 + e_2 = 0 \tag{25a}$$

where

$$a_2 = \left[-32\eta_m + 136\eta_m^2 - 256\eta_m^3 + 292\eta_m^2 - 224\eta_m^5 + 112\eta_m^6 - 32\eta_m^7 + 4\eta_m^8\right] R_4 S_4$$

$$+\left[-4 + 24\eta_m^2 - 32\eta_m^3 + 12\eta_m^4\right] R_4 V_4 + \left[-32\eta_m + 72\eta_m^2 \text{-} 64\eta_m^3 + 20\eta_m^4\right] S_4 V_4 - 4V_4^2 \tag{25b}$$

$$b_2 = \left[-8\eta_m + 28\eta_m^2 - 36\eta_m^3 + 20\eta_m^4 - 4\eta_m^5\right] R_4 S_1 + \left[-8\eta_m + 10\eta_m^2 - 10\eta^3 + 5\eta_m^4 - \eta_m^5\right] R_1 S_4$$

$$+\left[-4 + 12\eta_m - 12\eta_m^2 + 4\eta_m^3\right] R_4 V_1 + \left[-8\eta_m + 18\eta_m^2 - 16\eta_m^3 + 5\eta_m^4\right] S_4 V_1$$

$$-(3\eta_m + 1) R_1 V_4 - 8\eta_m S_1 V_4 - 5V_1 V_4 \tag{25c}$$

$$c_2 = \left[4 - 16\eta_m + 24\eta_m^2 - 16\eta_m^3 + 4\eta_m^4\right] R_4 + 4V_4 \tag{25d}$$

$$d_2 = \left[-2\eta_m + \eta_m^2\right] R_1 S_1 - 2\eta_m S_1 V_1 - R_1 V_1 - V_1^2 \tag{25e}$$

$$e_2 = (1 - \eta_m) R_1 + V_1 \tag{25f}$$

Equations (23) and (25) may be solved numerically to obtain the efficiency at maximum power conditions once the constant parameters are assigned specific values. Such a numerical solution is given here for several combinations of the parameters involved and results and displayed graphically in Figures 2–4.

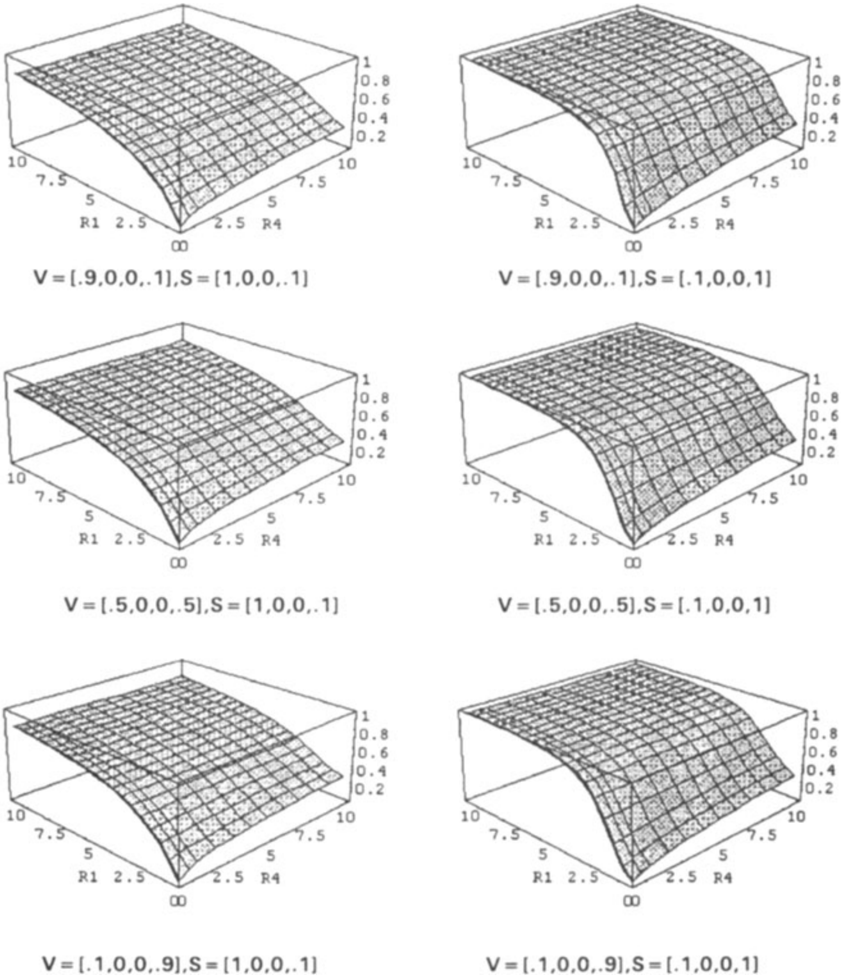

Figure 2. Variation of the efficiency at maximum power with R_1 and R_4 for $T_2/T_1 = 0$

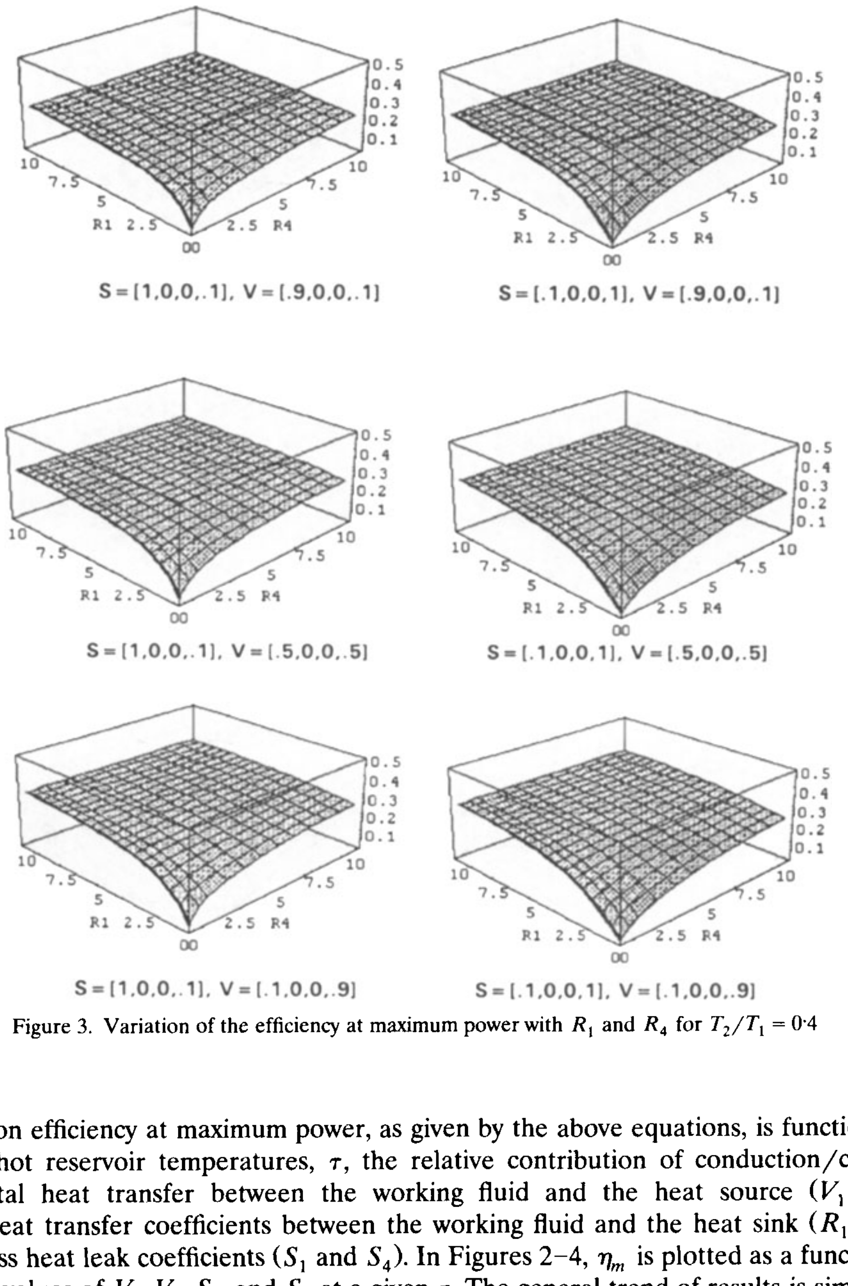

Figure 3. Variation of the efficiency at maximum power with R_1 and R_4 for $T_2/T_1 = 0{\cdot}4$

The conversion efficiency at maximum power, as given by the above equations, is function of the ratio of the cold to hot reservoir temperatures, τ, the relative contribution of conduction/convection and radiation to total heat transfer between the working fluid and the heat source (V_1 and V_4), the dimensionless heat transfer coefficients between the working fluid and the heat sink (R_1 and R_4), and the dimensionless heat leak coefficients (S_1 and S_4). In Figures 2–4, η_m is plotted as a function of R_1 and R_4 for different values of V_1, V_4, S_1, and S_4 at a given τ. The general trend of results is similar and shows η_m, for constant values of τ, to increase with R_1 and R_4 for given V_1, V_4, S_1 and S_4 and to decrease with increasing τ for given values of the various parameters involved. This is to be expected since, when S_1, S_4, and τ are constant, increasing R_1 and R_4 (the dimensionless heat transfer coefficients) reflects an increase in the heat transfer coefficients or a decrease in resistance to heat flow between the working fluid and the heat sink and results in a lower temperature for the cold reservoir. This, in turn, causes the Carnot-like engine to operate between a hot and cold reservoirs of higher temperature difference and consequently, results in an increase of its efficiency. Furthermore, at constant values of R_1, R_4, S_1, and S_4, an increase in τ produces closer hot and cold reservoir temperatures and hence a less efficient engine.

By comparing results in Figures 2–4, it can easily be inferred that, the efficiency at maximum power

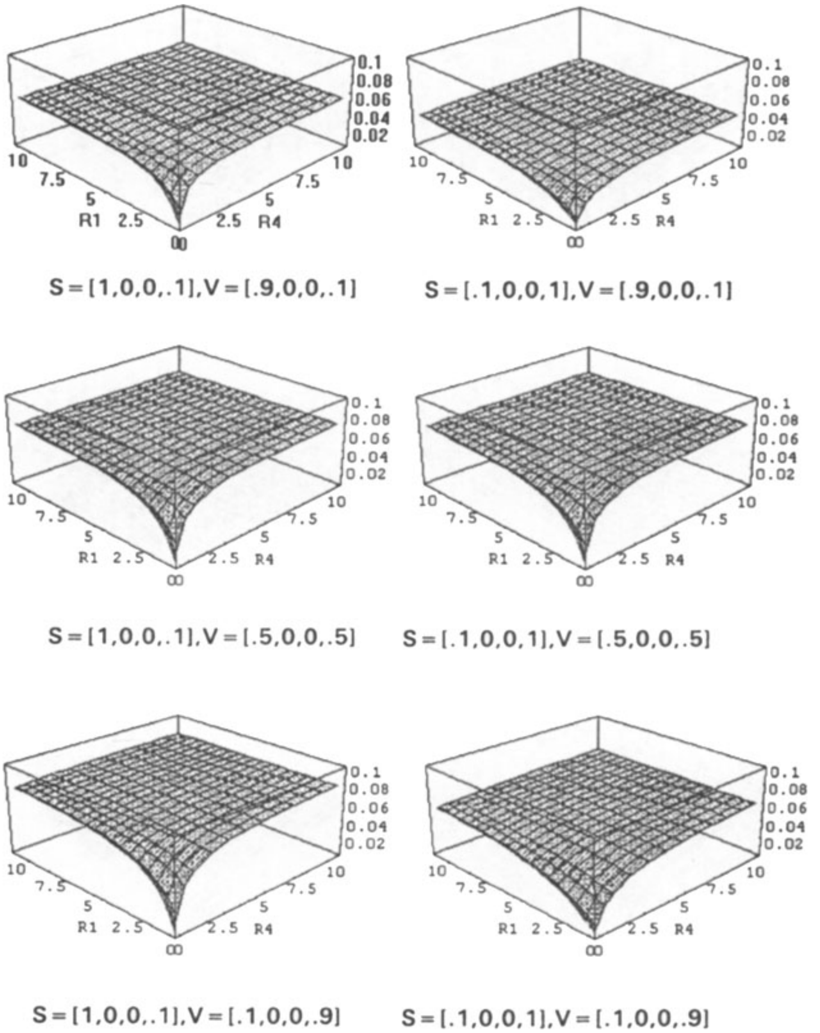

Figure 4. Variation of the efficiency at maximum power with R_1 and R_4 for $T_2/T_1 = 0{\cdot}8$

conditions of the endoreversible engine at given R_1 and R_4 increases with increasing the dimensionless radiation heat transfer coefficients, i.e. when V_4 and/or S_4 increase. This is so because as V_4 and S_4 increase, the contribution of radiation heat transfer increases implying higher hot reservoir temperature and thereby higher efficiency.

Derivation of previous reported work from the new model

In this section, the work reported in several references is shown to represent very special cases of the general formulation presented here. This is done by starting with the most general formulation in the literature (Moukalled *et al.*, 1995) down to the original Curzon–Ahlborn efficiency (Curzon and Ahlborn, 1975).

The results reported in Moukalled *et al.* (1995) may be generated from the current general model by setting $n_1 = n_2 = n$, $m_1, = m_2 = m$, $l_1 = l_2 = l$, $V_i = 1$, $R_i = R$, and $S_i = S$. Performing this step, equations (14), (15), (13), (16), and (17) are transformed respectively to

$$\frac{W}{\alpha T_1^n} = 1 + R\tau^m - t_1^n - R\,t_1^m(1-\eta)^m \tag{26}$$

$$\frac{dt_1}{d\eta} = \frac{m(1-\eta)^{m-1}t_1}{\left[\frac{n}{R}t_1^{n-m} + m(1-\eta)^m\right]} \tag{27}$$

$$(1-\eta)t_1^n + R(1-\eta)^m t_1^m - S\left[\eta + (1-\eta)^{l+1} - (1-\eta)^l\right]t_1^l - (1-\eta) - R\tau^m = 0 \tag{28}$$

$$\frac{dt_1}{d\eta} = \frac{t_1^n + mR(1-\eta)^{m-1}t_1^m + S\left[1-(l+1)(1-\eta)^l + l(1-\eta)^{l-1}\right]t_1^l - 1}{n(1-\eta)t_1^{n-1} + mR(1-\eta)^m t_1^{m-1} - S\left[\eta + (1-\eta)^{l+1} - (1-\eta)^l\right]lt_1^{l-1}} \tag{29}$$

$$\begin{aligned} &m(1-\eta_m)^{m-1}\left\{n(1-\eta_m)t_1^n + mR(1-\eta_m)^m t_1^m - S\left[\eta_m + (1-\eta_m)^{l+1} - (1-\eta_m)^l\right]lt_1^l\right\} \\ &\quad = \left[\frac{n}{R}t_1^{n-m} + m(1-\eta_m)^m\right]\left\{t_1^n + mR(1-\eta_m)^{m-1}t_1^m + S\left[1-(l+1)(1-\eta_m)^l + l(1-\eta_m)^{l-1}\right]t_1^l - 1\right\} \end{aligned} \tag{30}$$

which are the equations presented in Moukalled *et al.* (1995).

Since, based on the model presented in Moukalled *et al.* (1995) which is a special case of the general model presented here, most of the results reported in the literature were derived, it is obvious that the same result may be obtained here. Moreover, the derivations being reported in Monkalled *et al.* (1995), there is no need to elaborate on them here and only a brief description of the final results along with the respective values of the various parameters involved is given next.

The general case $n = m = l$

Many of the well known formulae in the literature may be derived from the general model for the case when $n = m = l$ whereby the same mode of heat transfer governs all heat exchange processes. In this situation, the above nonlinear system of equations may be reduced analytically to a single equation in η_m which may always be solved numerically. However, depending on the values of R, S, and τ, analytical solutions may sometimes be possible. Under these conditions, the solution to the above system of equations gives the following explicit relations for t_1 and η_m:

$$t_1 = \left[-\frac{a_1}{b_1}\right]^{1/n} \tag{31}$$

where

$$a_1 = n\left[\frac{1}{R} + (1-\eta_m)^n\right] \tag{32}$$

$$\begin{aligned} b_1 = {} & n^2\left[(1-\eta_m)^n + R(1-\eta_m)^{2n-1}\right] - n\left[\frac{1}{R} + (1-\eta_m)^n\right]\left[1 + nR(1-\eta_m)^{n-1}\right] \\ & + n^2 S(1-\eta_m)^{n-1}\left[-1 + (1-\eta_m) - (1-\eta_m)^{n+1} + (1-\eta_m)^n\right] \\ & - nS\left[\frac{1}{R} + (1-\eta_m)^n\right]\left[1 - (n+1)(1-\eta_m)^n + n(1-\eta_m)^{n-1}\right] \end{aligned} \tag{33}$$

and

$$\begin{aligned} & R[R + S(1+\tau^n R)](1-\eta_m)^{2n} + n(S + R + RS)(1-\eta_m)^{n+1} \\ & \quad + \{(1-n)[(R+S) - R^2\tau^n(1+S)] - RS(1+n)(1-\tau^n)\}(1-\eta_m)^n \\ & \quad - n\tau^n R[R + RS + S](1-\eta_m)^{n-1} - (S + R\tau^n + RS\tau^n) = 0 \end{aligned} \tag{34}$$

Having derived the general equation that η_m should satisfy, the exponent n in equation (34) is assigned, consecutively, the values -1, 1 and 4. These chosen exponents, describe the well-known Fourier, Newton, and radiative heat transfer laws. As shown next, many established formulae derived in several references are easily obtained here as a very special case of the general formulation.

Case 1: $n = m = l = -1$:
The heat transfer law governing all heat exchange processes in Fourier's law used in irreversible thermodynamics. Substituting n by -1, equation (34) reduces to:

$$[R(1+\tau)+S(2\tau+R+\tau R)](1-\eta_m)^2-2[R(\tau-R)+S(\tau-R^2)](1-\eta_m)-R^2(1+\tau)-SR(2R+1+\tau)=0 \tag{35}$$

This quadratic equation has the general solution:

$$\eta_m = 1-\frac{[R(\tau-R)+S(\tau-R^2)]}{R(1+\tau)+S(2\tau+R+\tau R)} - \frac{\sqrt{[R(\tau-R)+S(\tau-R^2)]^2+R[R(1+\tau)+S(2\tau+R+\tau R)][R(1+\tau)+S(1+\tau+2R)]}}{R(1+\tau)+S(2\tau+R+\tau R)} \tag{36}$$

for given values of R, S, and τ. If there is no heat leak ($S = 0$) and $R \to \infty$, then:

$$\lim_{R\to\infty} \eta_m = \frac{1}{2}(1-\tau) \tag{37}$$

This is the result obtained by Chen and Yan (1989) and DeVos (1985) when the only thermal resistance is between the working fluid and the high temperature source. Additionally, if $S = 0$ and $R = 0$ the efficiency at maximum power, as found by Chen and Yan (1989), is given by:

$$\lim_{R\to 0} \eta_m = \frac{1-\tau}{1+\tau} \tag{38}$$

The above equation is valid when the only thermal resistance is between the working fluid and the low temperature source.

Case 2: $n = m = l = 1$:
Using $n = 1$ in equation (1) and (34), then equation (1) expresses Newton's law and the relation for η_m reduces to:

$$[R^2(1+S\tau)+R(1+2S)+S]\eta_m^2-2[R^2(1+S\tau)+R(1+S+S\tau)+S]\eta_m+R(1+R)(1-\tau)=0 \tag{39}$$

The above equation for η_m has always the following closed form solution:

$$\eta_m = 1-\frac{(1-\tau)RS}{R^2(1+\tau S)+R(1+2S)+S} - \sqrt{\left[1-\frac{(1-\tau)RS}{R^2(1+\tau S)+R(1+2S)+S}\right]^2-\frac{R(1+R)(1-\tau)}{R^2(1+\tau S)+R(1+2S)+S}} \tag{40}$$

It is interesting to note that when there is no leakage ($S = 0$), equation (40) above, irrespective of the value of R, reduces to:

$$\eta_m = 1-\sqrt{\tau} \tag{41}$$

which is the Curzon–Ahlborn efficiency (Curzon and Ahlborn, 1975). Thus, equation (40) is a generalized form of that efficiency.

Case 3: $n = m = l = 4$:
For this case, all heat transfer processes including the heat leak, take place through a radiative heat transfer mode. Upon inserting the respective powers into equations (31) and (34), the following relations for t_1 and η_m are obtained:

$$t_1 = \left[\frac{(1-\eta_m)+\tau^4 R}{(R+\eta_m S)(1-\eta_m)^4+(1-\eta_m)-\eta_m S}\right]^{1/4} \tag{42}$$

and

$$R[S + R + \tau^4 SR](1 - \eta_m)^8 + 4[RS + R + S](1 - \eta_m)^5$$
$$- [3S + 5RS + 3R - 3SR^2\tau^4 - 5RS\tau^4 - 3R^2\tau^4](1 - \eta_m)^4$$
$$- 4R[RS + R + S]\tau^4(1 - \eta_m)^3 - S - R\tau^4 - RS\tau^4 = 0 \quad (43)$$

The highest possible efficiency is attained when there is no leakage and when $\tau = 0$. If this is the case, then equation (43) may be written as:

$$R(1 - \eta_m)^4 + 4(1 - \eta_m) - 3 = 0 \quad (44)$$

This equation reduces to that of DeVos (1991) when $R \to 1$, i.e.,

$$\eta_m^4 - 4\eta_m^3 + 6\eta_m^2 - 8\eta_m + 2 = 0 \quad (45)$$

and the solution gives $\eta_m = 0{\cdot}307$. Equation (44), is, however, the more general one in that the variation of efficiency at maximum power with R is shown.

Finally, of interest is the case for which $S = 0$ and $R \to \infty$. Substitution of these values into equations (42), and (43) results in the following equations for the efficiency at maximum power and t_{01}:

$$\eta_m = 1 - \frac{T_2}{t_{01}} \quad (46)$$

and

$$4t_{01}^5 - 3t_{01}^4 T_2 - T_1^4 T_2 = 0 \quad (47)$$

The last equation, known as the Castans relation (Chen and Yan, 1989), is a practical formula in solar energy conversion systems and shows again that the results of this paper are the most general.

CONCLUSION

A new generalized mathematical model for studying the performance of endoreversible heat engines with heat leak was presented. The model was employed for performance of Carnot-like engines under combined modes of heat supply, release, and leak. Several new equations were presented and many well-known ones, developed in several references, were found to represent very special cases of the general formulation. The model can be applied to predicting the performance of a variety of energy conversion systems of practical interest.

ACKNOWLEDGEMENT

The financial support provided by the University Research Board of the American University of Beirut through Grant No. 113040-48816 is gratefully acknowledged.

a_1, a_2	= coefficients in algebraic equation
b_1, b_2	= coefficients in algebraic equation
c_1, c_2	= coefficients in algebraic equation
d_2	= coefficient in algebraic equation
e_2	= coefficient in algebraic equation
l, l_1, l_2	= integers representing the heat transfer mode
m, m_1, m_2	= integers representing the heat transfer mode
n, n_1, n_2	= integers representing the heat transfer mode

Q_1	= heat entering the endoreversible heat engine
Q_2	= heat leaving the endoreversible heat engine
R, R_i	= dimensionless parameters
S, S_i	= dimensionless parameters
t_{01}	= dimensional temperature of the working material entering the engine
t_{02}	= dimensional temperature of the working material leaving the engine
T_1	= dimensional temperature of hot reservoir
t_1	= dimensional temperature of the working material entering the engine
T_2	= dimensional temperature of cold reservoir
V, V_1	= dimensionless parameters
W	= power output
α_i, β_i, γ_i	= heat transfer coefficients
η	= efficiency
η_m	= efficiency at maximum power
τ	= ratio of cold to hot reservoir temperatures

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