A Unified Formulation of the Segregated Class of Algorithms for Multi-Fluid Flow at All Speeds

Plan of Presentation

- Single-Fluid Flows
- Governing Equations
- Discretized Form
- Solution Algorithms
- Collocated Arrangement
- All-Speed Flows
- Multi-Fluid Flows
- Governing Equations
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- Solution Algorithm
- Phase 1: Pressure Equation
- Geometric Conservation Based Algorithms
- Mass Conservation Based Algorithms
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Governing Equations for Single Fluid

Momentum Conservation

$$\frac{\partial(\rho \mathrm{v})}{\partial t} + \nabla \cdot (\rho \mathrm{v}\mathrm{v}) = -\nabla P + \nabla \cdot (\mu \nabla \mathrm{v}) + \frac{1}{3}\nabla(\mu \nabla \cdot \mathrm{v})$$

Energy Conservation

$$\frac{\partial(\rho T)}{\partial t} + \nabla .(\rho \mathrm{v} T) = \frac{1}{c_p}\left\{\nabla \cdot (k \nabla T) + \beta T\left[\frac{\partial P}{\partial t} + \nabla \cdot (P \mathrm{v}) - P \nabla \cdot (\mathrm{v})\right] + \Phi + \dot{q}\right\}$$

Mass Conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathrm{v}) = 0$$

Auxiliary Relations

$$\rho = \frac{P}{RT} = C_\rho P$$

General Conservation Equation

$$\frac{\partial(\rho \phi)}{\partial t} + \nabla \cdot (\rho \mathrm{v} \phi) = \nabla \cdot \left(\Gamma^{\phi} \nabla \phi\right) + Q^{\phi}$$

Discretized Form

General Conservation Equation

$$\frac{(\rho\phi)_P}{\Delta t}\Omega + \sum_f \left(\rho \mathbf{v}\phi - \Gamma^{\phi}\nabla\phi\right)_f \cdot \mathbf{S_f} = Q^{\phi}\Omega + \frac{(\rho\phi)^{Old}}{\Delta t}\Omega$$

$$a_P^{\phi}\phi_P = \sum_{NB(P)} a_{NB}^{\phi}\phi_{NB} + b_P^{\phi}$$

$$\phi_P = \frac{\sum\limits_{NB(P)} a_{NB}^{\phi}\phi_{NB} + b_P^{\phi}}{a_P^{\phi}} = \mathrm{H}[\phi]_P$$

Momentum Conservation

$$\mathrm{v}_P - \mathrm{H}[\mathrm{v}]_P = -\mathrm{D}_P(\nabla P)_P$$

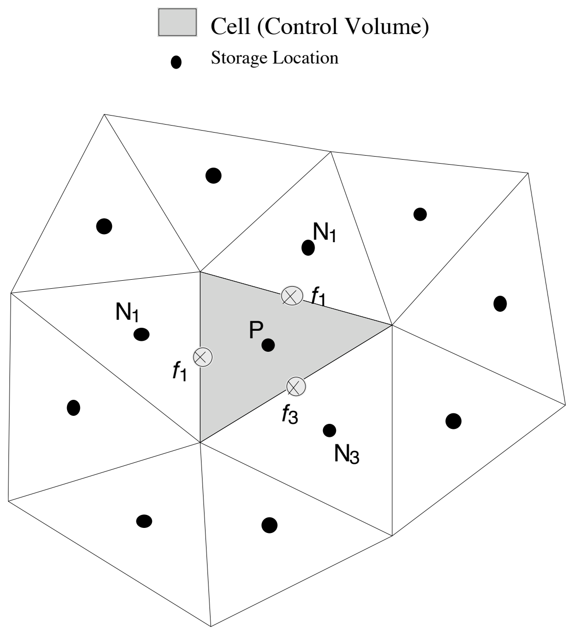
Mass Conservation

$$\frac{\left(\rho_P - \rho_P^{Old}\right)}{\delta t}\Omega + \sum_{\mathrm{f}} (\rho \mathbf{v}.\mathbf{S})_f = 0$$

$$\mathbf{v}_f - \overline{\mathrm{H}[\mathbf{v}]_f} = -\overline{\mathrm{D}}_f(\nabla P)_f$$

$$\mathbf{v}_f - \overline{\mathbf{v}_f} = -\overline{\mathrm{D}}_f\left(\nabla P - \overline{\nabla P}\right)_f$$ Rhi-Chow interpolation

$$\mathbf{v}_f - \overline{\mathrm{H}[\mathbf{v}]_f} = -\overline{\mathrm{D}}_f(\nabla P)_f$$



Pressure Correction Equation

$$\frac{\left(\rho_P-\rho_P^{Old}\right)}{\delta t}\Omega+\sum_{\mathrm{f}}(\rho\mathbf{v}.\mathbf{S})_f=0$$

$$\begin{cases} P=P^{(n)}+P' \\ \mathrm{v}=\mathrm{v}^*+\mathrm{v}' \qquad \left(u=u^*+u', v=v^*+v'\right) \\ \rho=\rho^{(n)}+\rho' \end{cases} \Longrightarrow \begin{array}{l} \mathbf{v}_P^*-\mathrm{H}\left[\mathbf{v}^*\right]_P=-\mathbf{D}_\mathbf{P}\left(\nabla P^{(n)}\right)_P \\ \mathbf{v}_\mathrm{P}-\mathrm{H}[\mathbf{v}]_P=-\mathrm{D}_P(\nabla P)_P \\ \mathbf{v}_P'-\mathrm{H}[\mathbf{v}']_P=-\mathrm{D}_P(\nabla P')_P \end{array} \quad \Rightarrow \mathbf{v}_f'-\overline{\mathrm{H}[\mathbf{v}']_f}=-\overline{\mathrm{D}}_f(\nabla P')_f$$

$$\frac{\left(\rho_P+\rho_P'-\rho_P^{Old}\right)}{\delta t}\Omega+\sum_{\mathrm{f}}\left(\rho_f^*\mathbf{v}_f^*+\rho_f^*\mathbf{v}_f'+\rho_f'\mathbf{v}_f^*+\rho_f'\mathbf{v}_f'\right).\mathrm{S}_\mathrm{f}=0$$

$$\frac{\Omega}{\Delta t}(\rho_P')+\sum_f\left(\rho_f'\mathbf{v}_f^*+\rho_f^*\mathbf{v}_\mathrm{f}'\right).\mathrm{S}=-\frac{\Omega}{\Delta t}\left(\rho_P^*-\rho_P^o\right)-\sum_f\left(\rho_f^*\mathbf{v}_f^*\cdot\mathbf{S}_f\right)-\sum_f\left(\rho_f'\mathbf{v}_f'\cdot\mathbf{S}_f\right)$$

$$\frac{\Omega}{\Delta t}(\rho_P')+\sum_f\left[U_f^*\rho_f'+\rho_f^*\left(\overline{\mathrm{H}[\mathbf{v}']_f}-\overline{\mathrm{D}}_f(\nabla P')_f\right)\cdot\mathbf{S}_f\right]=-\frac{\Omega}{\Delta t}\left(\rho_P^*-\rho_P^o\right)-\sum_f\left(\rho_f^*U_f^*\right)-\sum_f\left(\rho_f'\mathbf{v}_f'\cdot\mathbf{S}_f\right)$$

$$\boxed{\frac{\Omega C_\rho}{\Delta t}(P_P')+\sum_f\left(C_\rho U_f^*P'\right)_f-\sum_f\left(\rho_f^*\overline{\mathrm{D}}_f(\nabla P')_f\cdot\mathbf{S}_f\right)=-\left[\frac{\Omega}{\Delta t}\left(\rho_P^*-\rho_P^o\right)+\sum_f\left(\rho_f^*U_f^*\right)\right]-\sum_f\left(\rho_f'\mathbf{v}_f'\cdot\mathbf{S}_f\right)-\sum_f\left(\rho_f^*\overline{\mathrm{H}[\mathbf{v}']_f}\cdot\mathbf{S}_f\right)}$$

Pressure Correction Equation

Solution Algorithm

$$\frac{\Omega C_\rho}{\Delta t}(P'_P) + \sum_f \left(C_\rho U_f^* P'\right)_f - \sum_f \left(\rho_f^* \overline{\mathrm{D}}_f (\nabla P')_f \cdot \mathbf{S}_f\right) = -\left[\frac{\Omega}{\Delta t}\left(\rho_P^* - \rho_P^o\right) + \sum_f \underbrace{\left(\rho_f^* U_f^*\right)}_{\text{High Resolution}}\right] - \sum_f \underbrace{\left(\rho'_f \mathbf{v}'_f \cdot \mathbf{S}_f\right)}_{\text{Neglect}} - \sum_f \left(\rho_f^* \overline{\mathrm{H}[\mathbf{v}']_f} \cdot \mathbf{S}_f\right)$$

$\mathbf{v}'_P = \mathrm{H}[\mathbf{v}']_P - \mathrm{D}_P(\nabla P')_P$ Neglect: $\mathrm{H}[\mathbf{v}']_P$ $\Rightarrow \mathbf{v}'_f = \overline{\mathrm{D}_f}(\nabla P')_f$ SIMPLE

$\mathbf{v}'_P = \mathrm{H}[\mathbf{v}']_P - \mathrm{D}_P(\nabla P')_P$

$$\left(1 - \mathrm{H}[1]_P\right)\mathbf{v}'_P = \mathrm{H}\left[\mathbf{v}' - \mathbf{v}'_P\right]_P - \mathrm{D}_P(\nabla P')_P$$

$$\mathbf{v}'_P = \frac{\mathrm{H}\left[\mathbf{v}' - \mathbf{v}'_P\right]_P}{\left(1 - \mathrm{H}[1]_P\right)} - \frac{\mathrm{D}_P}{\left(1 - \mathrm{H}[1]_P\right)}(\nabla P')_P$$

$\mathbf{v}'_P = \tilde{\mathrm{H}}[\mathbf{v}']_P - \tilde{\mathrm{D}}_P(\nabla P')_P$ Neglect: $\tilde{\mathrm{H}}[\mathbf{v}']_P$ $\Rightarrow \mathbf{v}'_f = \overline{\tilde{\mathrm{D}}}_f(\nabla P')_f$ SIMPLEC

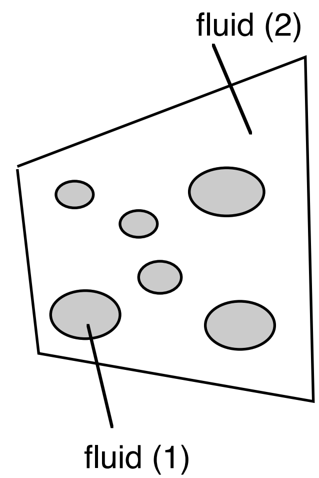
$\mathbf{v}'_P = \mathrm{H}[\mathbf{v}']_P - \mathrm{D}_P(\nabla P')_P = \mathrm{D}_P^{sx}(\nabla P')_P$ $\quad -\mathrm{D}_P^{sx}(\nabla P')_P = \mathrm{H}[-\mathrm{D}^{sx}(\nabla P')]_P - \mathrm{D}_P(\nabla P')_P$

$\mathrm{v}'_P = \mathrm{H}[\mathrm{v}']_P - \mathrm{D}_P(\nabla P')_P = -\mathrm{D}_P^{sx}(\nabla P')_P$ $\quad \Rightarrow \mathrm{D}_P^{sx}(\nabla P')_P = -\mathrm{H}[\mathrm{D}^{sx}(\nabla P')]_P + \mathrm{D}_P(\nabla P')_P$

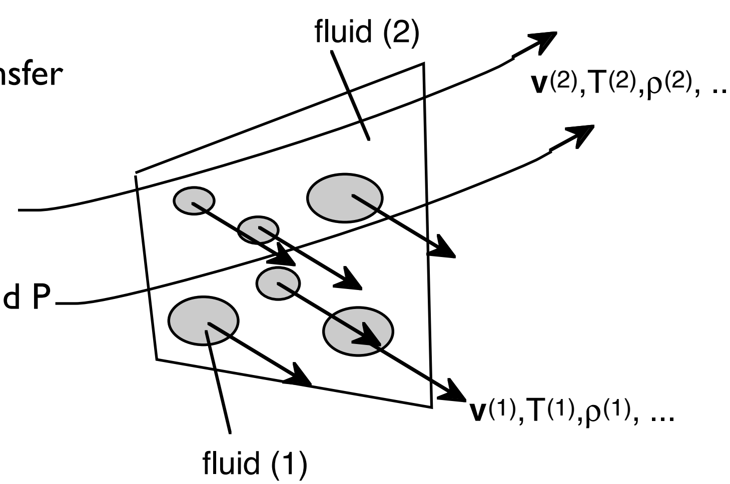
assume $\mathrm{H}\left[-\mathrm{D}^{sx}(\nabla P')\right]_P = (\nabla P')_P \mathrm{H}[-\mathrm{D}^{sx}]_P$ $\quad \Rightarrow \mathrm{D}_P^{sx} = -\mathrm{H}[\mathrm{D}^{sx}]_P + \mathrm{D}_P$ SIMPLEX

Multi-Fluid Flows: Assumptions

- Inter-Penetrating Continua
  - Each fluid (k) is assumed present in each control volume, and assigned a volume fraction $r^{(k)}$ equal to the fraction of the volume of the volume occupied
- Distinct Field Variables
  - Each fluid has its own field variables, velocity, pressure [$P^{(k)}$], temperature[$T^{(k)}$], density [$\rho^{(k)}$]etc…
    - $\rho^{(k)}$ = material density of fluid (k)
    - $\rho^{(k)}=r^{(k)}\ \rho^{(k)}$=effective density of fluid (k)
  - Velocities, temperature etc… coupled by inter-phase transfer models for momentum, heat etc...
- Common Pressure Field
  - Simplest multi-fluid models assume a shared pressure field P
    - $P^{(1)}=P^{(2)}= \ldots =P^{(k)}$



Volume of control cell = Ω
Volume occupied by fluid (1) = $r^{(1)}\ \Omega$
Volume occupied by fluid (2) = $r^{(2)}\ \Omega$



Governing Equations

Phasic Mass Conservation Equations

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\mathbf{u}^{(k)}\rho^{(k)}\right)=r^{(k)}\dot{M}^{(k)}$$

Continuity Form

$$\frac{\partial\left(\rho^{(k)}r^{(k)}\right)}{\partial t}+\nabla.\left(\rho^{(k)}\mathbf{u}^{(k)}r^{(k)}\right)=r^{(k)}\dot{M}^{(k)}$$

Volume Fraction Form

Global Mass Conservation Equations

$$\sum_k\left(\frac{\partial\left(r^{(k)}\rho^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)\right)=0$$

Phasic Momentum Equations

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\mathbf{u}^{(k)}\right)=\nabla.\left(r^{(k)}\mu^{(k)}\nabla\mathbf{u}^{(k)}\right)+r^{(k)}\left(-\nabla P+\mathbf{B}^{(k)}\right)+\mathbf{I}^{(k)}$$

$$\mathbf{I}^{(k)}=\sum_{m\neq k}g^{(km)}\left(\mathbf{u}^{(m)}-\mathbf{u}^{(k)}\right)$$

Phasic Energy Equations

$$\frac{\partial\left(r^{(k)}\rho^{(k)}T^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}T^{(k)}\right)=\frac{1}{c_P^{(k)}}\nabla.\left(r^{(k)}k^{(k)}\nabla T^{(k)}\right)$$
$$+\frac{r^{(k)}}{c_P^{(k)}}\left\{\beta^{(k)}T^{(k)}\left[\frac{\partial P}{\partial t}+\nabla.\left(P\mathbf{u}^{(k)}\right)-P\nabla.\left(\mathbf{u}^{(k)}\right)\right]+\Phi^{(k)}+\dot{q}^{(k)}\right\}$$

Auxiliary Relations

$$\sum_k r^{(k)}=1 \qquad \rho^{(k)}=\rho^{(k)}\left(P,T^{(k)}\right)$$

Discretized Form

General Conservation Equation

$$\frac{\left(r^{(k)}\rho^{(k)}\phi^{(k)}\right)_P}{\Delta t}\Omega+\sum_f\left(r^{(k)}\rho^{(k)}\mathbf{v}^{(k)}\phi^{(k)}-r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)}\right)_f\cdot\mathbf{S_f}=r^{(k)}Q^{(k)}\Omega+\frac{\left(r^{(k)}\rho^{(k)}\phi^{(k)}\right)^{Old}}{\Delta t}\Omega$$

$$a_P^{(k)}\phi_P^{(k)}=\sum_{NB(P)}a_{NB}^{(k)}\phi_{NB}^{(k)}+b_P^{(k)}$$

$$\phi_P^{(k)}=\frac{\sum_{NB(P)}a_{NB}^{(k)}\phi_{NB}^{(k)}+b_P^{(k)}}{a_P^{(k)}}=\mathrm{H}[\phi^{(k)}]_P$$

Momentum Conservation

$$\mathbf{A}_P^{(k)}\mathbf{u}_P^{(k)}=\sum_{NB}\mathbf{A}_{NB}^{(k)}\mathbf{u}_{NB}^{(k)}+\mathbf{B}_P^{(k)}-r_P^{(k)}\Omega_P\nabla_P(P)+\Omega_P\sum_{m\neq k}g^{(km)}\mathbf{u}_p^{(m)}$$

$$\mathbf{u}_P^{(k)}=\mathbf{HP}_P\left[\mathbf{u}^{(k)}\right]-r^{(k)}\mathbf{D}_P^{(k)}\nabla_P(P)$$

$$\mathbf{u}_P^{(k)}=\mathbf{HI}_P\left[\mathbf{u}^{(k)}\right]+\mathbf{D}_P^{(k)}\sum_{m\neq k}\left(g^{(km)}\mathbf{u}_P^{(m)}\right)$$

Mass Conservation

$$r_P^{(k)}=H_P\left[r^{(k)}\right]$$

$$\frac{\left(r_P^{(k)}\rho_P^{(k)}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega-\sum_f r_f^{(k)}\mathbf{u}_f^{(k)}.\mathbf{S}_f=r_P^{(k)}\dot{M}_P^{(k)}\Omega$$

Equations and Variables

For N-Fluid flow we have

Equations

3NF phasic momentum equations

NF phasic continuity

NF phasic energy equation

1 Auxiliary Geometric Conservation Relation

NF density-pressure relation

Total: 6NF+1

Variables

3NF velocity variables, $\mathbf{v}^{(k)}$

NF volume fraction variables, $r^{(k)}$

NF temperature variables, $T^{(k)}$

1 Pressure field, P

NF density fields, $\rho^{(k)}$

Total: 6NF+1

Geometric Conservation Based Algorithm

Equations	# Equations	Variables	# Variables
$\frac{\partial\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\mathbf{u}^{(k)}\right)=\nabla.\left(r^{(k)}\mu^{(k)}\nabla\mathbf{u}^{(k)}\right)+r^{(k)}\left(-\nabla P+\mathbf{B}^{(k)}\right)+\mathbf{I}^{(k)}$	3NF	$\mathbf{u}^{(k)}$	3NF
$\frac{\partial\left(r^{(k)}\rho^{(k)}T^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}T^{(k)}\right)=\frac{1}{c_P^{(k)}}\nabla.\left(r^{(k)}k^{(k)}\nabla T^{(k)}\right)+\frac{r^{(k)}}{c_P^{(k)}}\left\{\beta^{(k)}T^{(k)}\left[\frac{\partial P}{\partial t}+\nabla.\left(P\mathbf{u}^{(k)}\right)-P\nabla.\left(\mathbf{u}^{(k)}\right)\right]+\Phi^{(k)}+\dot{q}^{(k)}\right\}$	NF	$T^{(k)}$	NF
$\frac{\partial\left(\rho^{(k)}r^{(k)}\right)}{\partial t}+\nabla.\left(\rho^{(k)}\mathbf{u}^{(k)}r^{(k)}\right)=\dot{M}^{(k)}r^{(k)}$	NF	$r^{(k)}$	NF
$\rho^{(k)}=\rho\left(P,T^{(k)},r^{(k)}\right)$	NF	$\rho^{(k)}$	NF
$\sum_k r^{(k)}=1$	1	P	1

Mass Conservation Based Algorithm

Equations	# Equations	Variables	# Variables
$\frac{\partial\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\mathbf{u}^{(k)}\right)=\nabla.\left(r^{(k)}\mu^{(k)}\nabla\mathbf{u}^{(k)}\right)+r^{(k)}\left(-\nabla P+\mathbf{B}^{(k)}\right)+\mathbf{I}^{(k)}$	3NF	$\mathbf{u}^{(k)}$	3NF
$\frac{\partial\left(r^{(k)}\rho^{(k)}T^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}T^{(k)}\right)=\frac{1}{c_P^{(k)}}\nabla.\left(r^{(k)}k^{(k)}\nabla T^{(k)}\right)$ $+\frac{r^{(k)}}{c_P^{(k)}}\left\{\beta^{(k)}T^{(k)}\left[\frac{\partial P}{\partial t}+\nabla.\left(P\mathbf{u}^{(k)}\right)-P\nabla.\left(\mathbf{u}^{(k)}\right)\right]+\Phi^{(k)}+\dot{q}^{(k)}\right\}$	NF	$T^{(k)}$	NF
$\frac{\partial\left(\rho^{(k)}r^{(k)}\right)}{\partial t}+\nabla.\left(\rho^{(k)}\mathbf{u}^{(k)}r^{(k)}\right)=\dot{M}^{(k)}r^{(k)}$	NF-1	$r^{(k)}$	NF
$r^{(n)}=1-\sum_{k\neq n}r^{(k)}$	1		
$\sum_{k=1}^{NF}\left\{\frac{\partial\left(r^{(k)}\rho^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)\right\}=0$	1	P	1
$\rho^{(k)}=\rho\left(P,T^{(k)},r^{(k)}\right)$	NF	$\rho^{(k)}$	NF

PEA and SINCE for Inter-Fluid Coupling

PEA decouples the inter-fluid terms in the phasic momentum equations

$$\begin{aligned} \mathbf{u}_P^{(1)} &= \mathbf{HI}_P\left[\mathbf{u}^{(1)}\right] + \mathbf{D}_P^{(1)} g^{(12)} \mathbf{u}_P^{(2)} \\ \mathbf{u}_P^{(2)} &= \mathbf{HI}_P\left[\mathbf{u}^{(2)}\right] + \mathbf{D}_P^{(2)} g^{(21)} \mathbf{u}_P^{(1)} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_P^{(1)} &= \frac{\mathbf{HI}_P\left[\mathbf{u}^{(1)}\right] + \mathbf{D}_P^{(1)} g^{(12)} \mathbf{HI}_P\left[\mathbf{u}^{(2)}\right]}{1 - \mathbf{D}_P^{(1)} g^{(12)} \mathbf{D}_P^{(2)} g^{(21)}} \\ \mathbf{u}_P^{(2)} &= \frac{\mathbf{HI}_P\left[\mathbf{u}^{(2)}\right] + \mathbf{D}_P^{(2)} g^{(21)} \mathbf{HI}_P\left[\mathbf{u}^{(1)}\right]}{1 - \mathbf{D}_P^{(1)} g^{(12)} \mathbf{D}_P^{(2)} g^{(21)}} \end{aligned}$$

In SINCE, phasic and spatial coupling are accounted for in two separate steps

1

$$\begin{aligned} \breve{\mathbf{u}}_P^{(1)} &= \mathbf{D}_P^{(1)} \sum_{m \neq 1} g^{(1m)} \breve{\mathbf{u}}_P^{(m)} + \mathbf{HI}_P\left[\mathbf{u}^{(1)}\right] \\ \breve{\mathbf{u}}_P^{(2)} &= \mathbf{D}_P^{(2)} \sum_{m \neq 2} g^{(2m)} \breve{\mathbf{u}}_P^{(m)} + \mathbf{HI}_P\left[\mathbf{u}^{(2)}\right] \\ &\vdots \\ \breve{\mathbf{u}}_P^{(k)} &= \mathbf{D}_P^{(k)} \sum_{m \neq k} g^{(km)} \breve{\mathbf{u}}_P^{(m)} + \mathbf{HI}_P\left[\mathbf{u}^{(k)}\right] \end{aligned}$$

2

$$\mathbf{u}_P^{(k)*} = \mathbf{HI}_P\left[\mathbf{u}^{(k)*}\right] + \mathbf{D}_P^{(k)} \sum_{m \neq k} g^{(km)} \breve{\mathbf{u}}_P^{(m)}$$

MCBA: Pressure Correction Equation

The global continuity equation written as

$$\sum_k \left\{ \frac{\left(r_P^{(k)}\rho_P^{(k)}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \sum_f \left(r_f^{(k)}\rho_f^{(k)}\mathbf{u}_f^{(k)}.\mathbf{S}_f\right) \right\} = \sum_k r_P^{(k)}\dot{M}_P^{(k)}\Omega = 0$$

Is used as the starting point for deriving the pressure correction equation. Taking correction terms

$$\begin{aligned} P &= P^{\circ} + P' \\ \mathbf{u}^{(k)} &= \mathbf{u}^{(k)^*} + \mathbf{u}^{(k)'} \\ \rho^{(k)} &= \rho^{(k)^{\circ}} + \rho^{(k)'} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \mathbf{u}_P^{(k)'} &= \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)^{\circ}}\mathbf{D}_P^{(k)}\nabla_P P' \\ \mathbf{u}_P^{(k)*} &= \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} \\ \mathbf{u}_P^{(k)} &= \mathbf{HP}_P[\mathbf{u}^{(k)}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P \end{aligned}$$

And substituting, yields after some manipulations

$$\boxed{\sum_k \left\{ \frac{\Omega}{\Delta t} r_P^{(k)} C_\rho^{(k)} P_P' + \sum_f \left[r_f^{(k)} U_f^{(k)*} C_\rho^{(k)} P_f' \right] - \sum_f \left[r_f^{(k)}\rho_f^{(k)*}\left(r_f^{(k)}\mathbf{D}^{(k)}\nabla P'\right)_f .\mathbf{S}_f \right] \right\} = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)}\rho_P^{(k)*} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \sum_f \left[r_f^{(k)}\rho_f^{(k)*}U_f^{(k)*} \right] \\ + \sum_f \left[r_f^{(k)}\rho_f^{(k)*}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right).\mathbf{S} \right] + \sum_f \left[r_f^{(k)}\rho_f'^{(k)}\mathbf{u}_f'^{(k)}.\mathbf{S} \right] \end{array} \right\}}$$

MCBA: Solution Algorithms

$$\boxed{\sum_k \left\{ \frac{\Omega}{\Delta t} r_P^{(k)} C_\rho^{(k)} P_P' + \sum_f \left[r_f^{(k)} U_f^{(k)*} C_\rho^{(k)} P_f' \right] - \sum_f \left[r_f^{(k)} \rho_f^{(k)*} \left(r_f^{(k)} \mathbf{D}^{(k)} \nabla P' \right)_f . \mathbf{S}_f \right] \right\} = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)} \rho_P^{(k)*} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \sum\limits_f \left[r_f^{(k)} \rho_f^{(k)*} U_f^{(k)*} \right] \\ + \sum\limits_f \left[r_f^{(k)} \rho_f^{(k)*} \left(\mathbf{HP}\left[\mathbf{u}^{(\mathrm{k})\prime} \right] \right) . \mathbf{S} \right] + \sum\limits_f \left[r_f^{(k)} \rho_f'^{(k)} \mathbf{u}_f'^{(k)} . \mathbf{S} \right] \end{array} \right\}}$$

approximate

neglect

Solution algorithms approximation similar to those of single-fluid

Weighted Pressure Correction

The pressure correction equation being derived from the global conservation equation, the intention is to correct the velocity fields so as to drive the global residual error, which is equal to the sum of the phasic residuals, to zero i.e. $RESC^{(1)} + RESC^{(2)} + .. + RESC^{(n)} \rightarrow 0$.

$$\sum_k \left\{ \frac{\Omega}{\Delta t} r_P^{(k)} C_\rho^{(k)} P'_P + \sum_f \left[r_f^{(k)} U_f^{(k)*} C_\rho^{(k)} P'_f \right] - \sum_f \left[r_f^{(k)} \rho_f^{(k)*} \left(r_f^{(k)} \mathbf{D}^{(k)} \nabla P' \right)_f . \mathbf{S}_f \right] \right\} = -\sum_k \left\{ \frac{r_P^{(k)} \rho_P^{(k)*} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \sum_f \left[r_f^{(k)} \rho_f^{(k)*} U_f^{(k)*} \right] \right\}$$

$$= \sum_k RESC^{(k)}$$

In the presence of a relatively very high density fluid such that $\rho^{(n)} >> \rho^{(k)}$ for $k \neq n$, the residual error of the n^{th} fluid will be of a magnitude commensurate with the respective phase density, i.e. $RESC^{(n)}$ is expected to be much larger than $RESC^{(k)}$ for $k \neq n$.

Instead solve for

$$\sum_k \left\{ \frac{\left(r_P^{(k)} \rho_P^{(k)} / \underline{\rho}^{(k)} \right) - \left(r_P^{(k)} \rho_P^{(k)} / \underline{\rho}^{(k)} \right)^{Old}}{\Delta t} \Omega_P + \sum_f \left[r_f^{(k)} \left(\frac{\rho_f^{(k)}}{\underline{\rho}^{(k)}} \right) \mathbf{u}_f^{(k)} . \mathbf{S}_f \right] \right\} = 0$$

Mutual influence of Volume Fractions

NF-1 volume fractions are computed from the phasic mass conservation equations,

$$r_P^{(k)*} = H_P\left[r^{(k)*}\right]$$

and the last one from the geometric conservation equations

$$r_P^{(n)*} = 1 - \sum_{k \neq n} r_P^{(k)*}$$

A drawback of this procedure is that the volume fraction field of any phase will not be influenced by the volume fraction fields of other phases except when calculating the n^{th} phase.

An improvement on the above procedure, is to solve all continuity equations to obtain all volume fractions and then enforce the geometric conservation constraint on the resulting volume fraction values using the following equation

$$r^{(k)} = \frac{r^{(k)*}}{\sum_m r^{(m)*}} = \frac{H_P\left[r^{(k)*}\right]}{\sum_m r^{(m)*}} \quad for \quad k = 1, NF$$

Implicit Volume Fraction Equations

$$r_P^{(k)*} = H\left[r^{(k)*}\right] = \frac{\sum_{NB} A_{NB}^{(k)} r_{NB}^{(k)*} + B_P'^{(k)}}{A_P^{(k)}}$$

For a two-fluid flow, the sum is written as

$$r_P^{(1)*} + r_P^{(2)*} = 1 = \frac{\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P'^{(1)}}{A_P^{(1)}} + \frac{\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P'^{(2)}}{A_P^{(2)}} \quad \Longrightarrow \quad A_P^{(1)} = \frac{A_P^{(2)}\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P'^{(1)}\right) + A_P^{(1)}\left(\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P'^{(2)}\right)}{A_P^{(2)}}$$

Substituting is the fluidic volume fraction equation yields

$$r_P^{(1)} = \frac{\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P'^{(1)}\right)}{A_P^{(1)}\left(\frac{\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P'^{(1)}}{A_P^{(1)}} + \frac{\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P'^{(2)}}{A_P^{(2)}}\right)}$$

$$r_P^{(1)} = \frac{\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P'^{(1)}\right)}{A_P^{(1)}\left(r^{(1)} + \frac{-A_P^{(1)} r^{(1)} + \sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P'^{(1)}}{A_P^{(1)}} + r^{(2)} + \frac{-A_P^{(2)} r^{(2)} + \sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P'^{(2)}}{A_P^{(2)}}\right)}$$

$$= \frac{\left(\sum_{NB} a_{NB}^{(1)} r_{NB}^{(1)*} + b_P'^{(1)}\right)}{a_P^{(1)}\left(1 - \frac{RESR^{(1)}}{a_P^{(1)}} - \frac{RESR^{(2)}}{a_P^{(2)}}\right)}$$

For NF fluid we get

$$r_P^{(k)*} = \frac{\left(\sum_{NB} A_{NB}^{(k)} r_{NB}^{(k)*} + B_P'^{(k)}\right)}{A_P^{(k)}\left(1 - \sum_{m} \frac{RESR^{(m)}}{A_P^{(m)}}\right)}$$

Bounding the Volume Fractions

We still need to guarantee than the volume fraction are bounded individually by 0 and 1, during all stages of the solution.

An intermediate value of r is calculated, For any intermediate value of $r^{(k)} > 1-\varepsilon$, β is modified to the form $MAX(1-r^{(k)},\gamma)$ where ε and γ are set to 0.05 and 10^{-10}.

$$r_P^{(k)\text{int}} = \frac{\sum_{NB} A_{NB} r_{NB}^{(k)\circ} + B_P + \frac{(1-\beta)}{\beta} A_P r_P^{(k)\circ}}{\frac{A_P}{\beta}} v$$

$$\frac{A_P}{\beta} r_P^{(k)C} = \sum_{NB} A_{NB} r_{NB}^{(k)C} + B_P + \frac{(1-\beta)}{\beta} A_P r_P^{(k)\circ}$$

With this relaxation factor $r^{(k)}$ is now calculated implicitly from:

Conclusions