Discretization of the Convection Term

Chapter 12

Convection Term

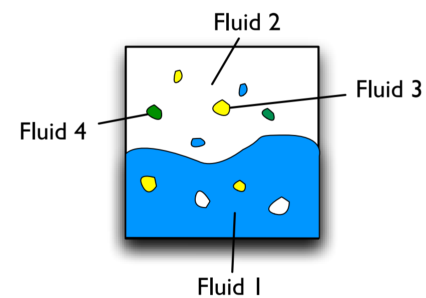
$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}\mathbf{v})}=\nabla\cdot\tau+\mathbf{B}$$

$$\frac{\partial(\rho E)}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}E)}=-\nabla\cdot(\mathbf{v}P)+\nabla\cdot(\mathbf{v}\cdot\tau)+Q$$

$$\frac{\partial(\rho\alpha)}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}\alpha)}=0$$

$$\frac{\partial(\rho\alpha C)}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}\alpha C)}=0$$

$$\boxed{\frac{\partial(\rho\phi)}{\partial t}+\nabla\cdot(\rho\mathbf{v}\phi)=\nabla\cdot(\Gamma\nabla\phi)+Q}$$

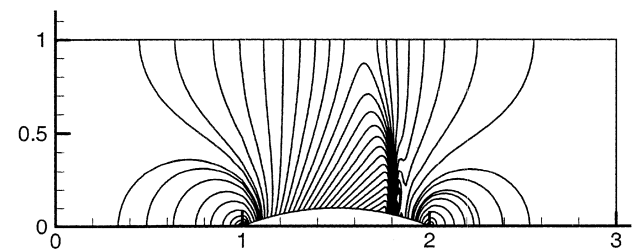


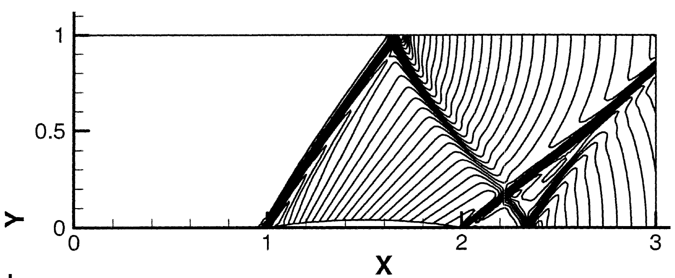
Pressure Correction Equation

Transient-like Term

$$\boxed{\frac{\Omega C_\rho}{\Delta t}(P'_P)}+\boxed{\sum_f\left(C_\rho U_f^* P'\right)_f}-\sum_f\left(\rho_f^*\overline{\mathrm{D}}_f(\nabla P')_f\cdot\mathbf{S}_f\right)=-\left[\boxed{\frac{\Omega}{\Delta t}\left(\rho_P^*-\rho_P^o\right)}+\sum_f\left(\rho_f^* U_f^*\right)\right]-\sum_f\left(\rho'_f\mathbf{v}'_f\cdot\mathbf{S}_f\right)-\sum_f\left(\rho_f^*\overline{\mathrm{H}[\mathbf{v}']_f}\cdot\mathbf{S}_f\right)$$

Advection-like Term
account for compressibility effects

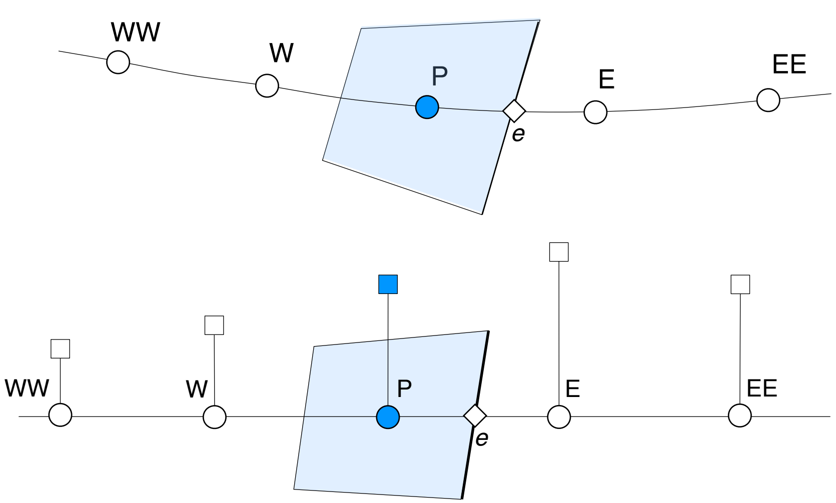


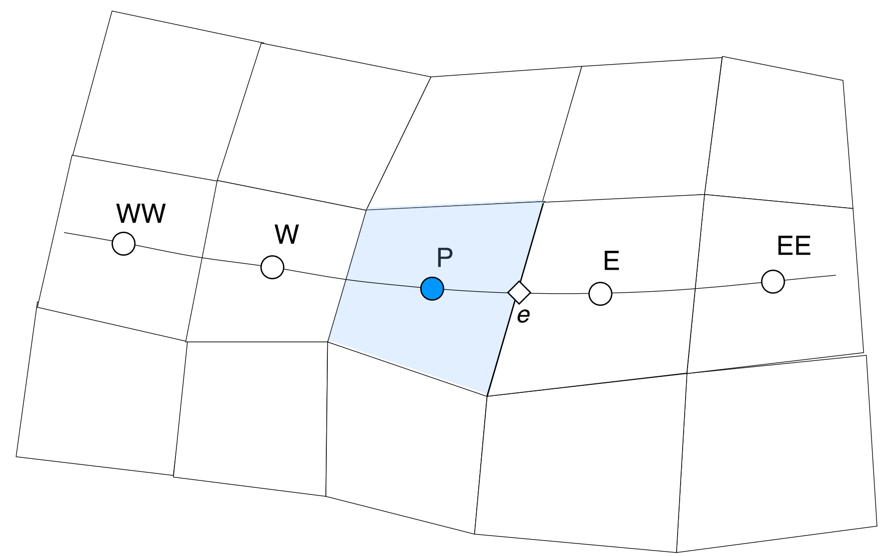


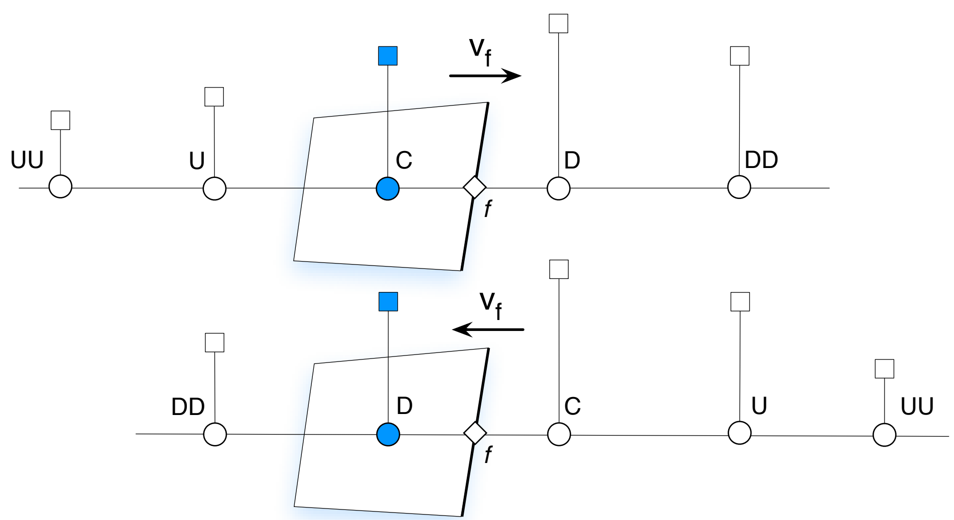
The simplicity of the convection term does not extend to its discretization (3 decades of research and continuing)

High Order Schemes

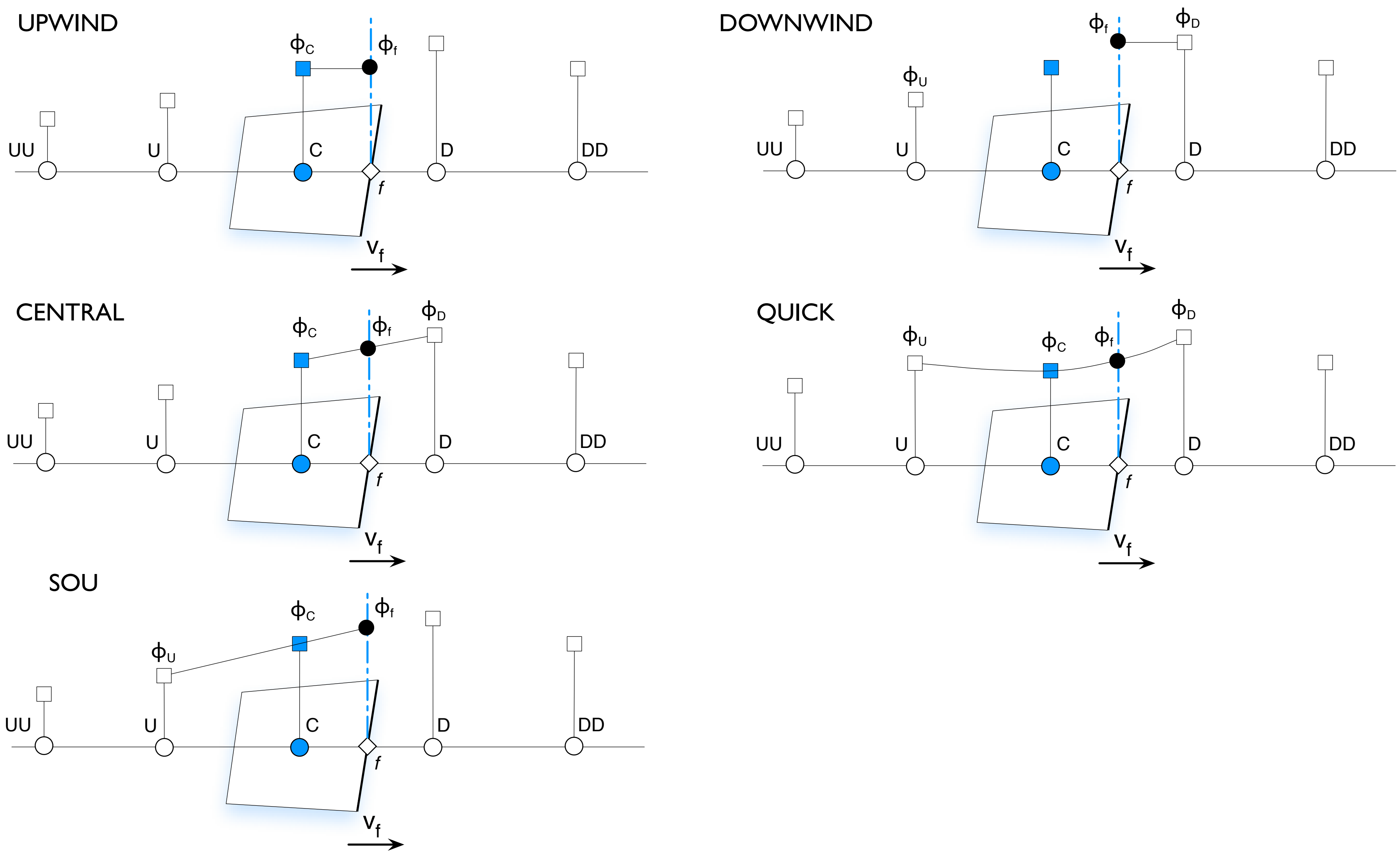
Notation







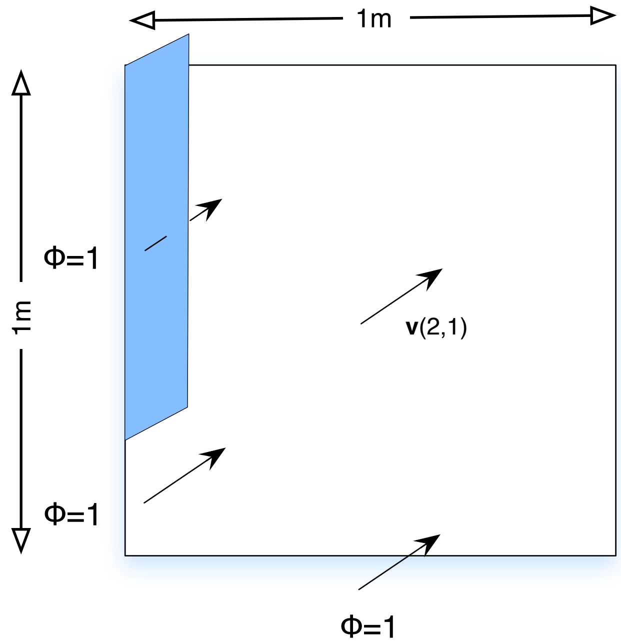
Convection Schemes

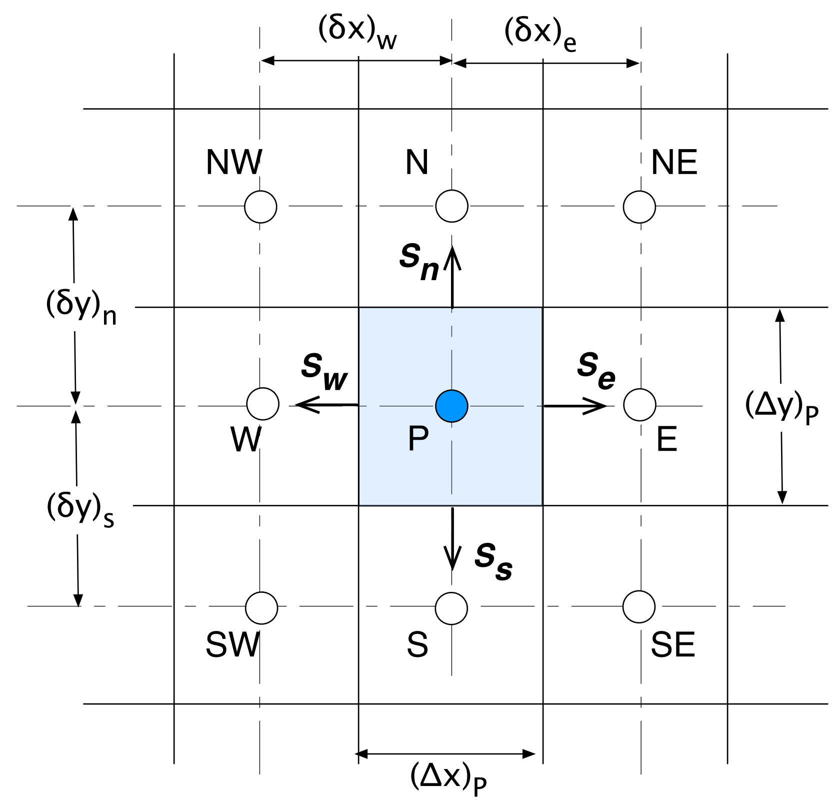


Convection Schemes

Convection Scheme	Formula
UPWIND	$\phi_f = \phi_C$
CENTRAL	$\phi_f = \dfrac{\phi_C + \phi_D}{2}$
SOU (Second Order Upwind)	$\phi_f = \dfrac{3}{2}\phi_C - \dfrac{1}{2}\phi_U$
QUICK (Quadratic Upwind Interpolation	$\phi_f = \dfrac{3}{4}\phi_C + \dfrac{3}{8}\phi_U - \dfrac{1}{8}\phi_D$
DOWNWIND	$\phi_f = \phi_D$

Convection Discretization





$$\nabla \cdot (\rho \mathbf{v} \phi) = 0 \qquad \sum_{f=nb(P)} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} = 0$$

$$(\rho_e \mathbf{v}_e \cdot \mathbf{S}_e)\phi_e + (\rho_w \mathbf{v}_w \cdot \mathbf{S}_w)\phi_w + (\rho_n \mathbf{v}_n \cdot \mathbf{S}_n)\phi_n + (\rho_s \mathbf{v}_s \cdot \mathbf{S}_s)\phi_s = 0$$

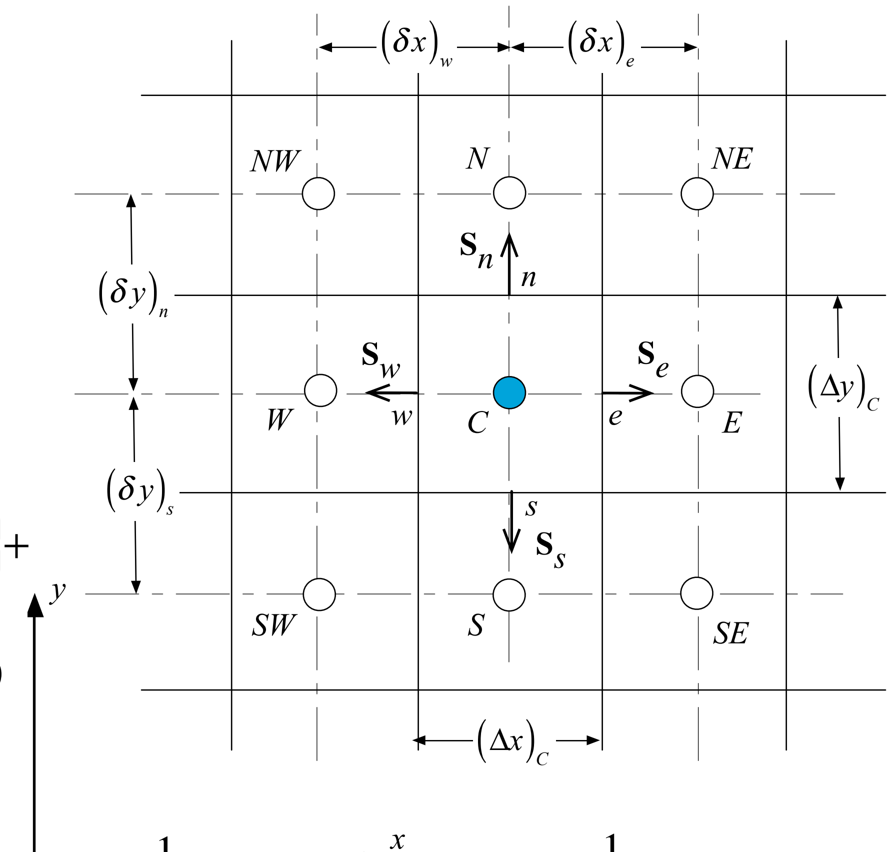
Second -Order Upwind

$$\phi_{f,SOU} = \left(\frac{3}{2}\phi_C - \frac{1}{2}\phi_U\right)$$

$$(\rho_e \mathbf{v}_e \cdot \mathbf{S}_e)\phi_e = (\rho_e \mathbf{v}_e \cdot \mathbf{S}_e)\left(\frac{3}{2}\phi_P - \frac{1}{2}\phi_W\right)$$

$$(\rho_w \mathbf{v}_w \cdot \mathbf{S}_w)\phi_w = (\rho_w \mathbf{v}_w \cdot \mathbf{S}_w)\left(\frac{3}{2}\phi_W - \frac{1}{2}\phi_{WW}\right)$$

$$(\rho_e \mathbf{v}_e \cdot \mathbf{S}_e)\left(\frac{3}{2}\phi_P - \frac{1}{2}\phi_W\right) + (\rho_w \mathbf{v}_w \cdot \mathbf{S}_w)\left(\frac{3}{2}\phi_W - \frac{1}{2}\phi_{WW}\right) +$$
$$(\rho_n \mathbf{v}_n \cdot \mathbf{S}_n)\left(\frac{3}{2}\phi_P - \frac{1}{2}\phi_S\right) + (\rho_s \mathbf{v}_s \cdot \mathbf{S}_s)\left(\frac{3}{2}\phi_S - \frac{1}{2}\phi_{SS}\right) = 0$$



$$a_P \phi_P + \sum_{N=E,W,N,S,WW,SS,NN,SS} a_N \phi_N = 0$$

$$a_W = \frac{3}{2}(\rho_w \mathbf{v}_w \cdot \mathbf{S}_w) - \frac{1}{2}(\rho_e \mathbf{v}_e \cdot \mathbf{S}_e) \qquad a_{WW} = -\frac{1}{2}(\rho_w \mathbf{v}_w \cdot \mathbf{S}_w)$$

$$a_S = \frac{3}{2}(\rho_s \mathbf{v}_s \cdot \mathbf{S}_s) - \frac{1}{2}(\rho_n \mathbf{v}_n \cdot \mathbf{S}_n) \qquad a_{SS} = -\frac{1}{2}(\rho_s \mathbf{v}_s \cdot \mathbf{S}_s)$$

$$a_P = -\sum_{N=E,W,N,S,EE,WW,NN,SS} a_N$$

$$a_E = a_N = a_{EE} = a_{NN} = 0$$

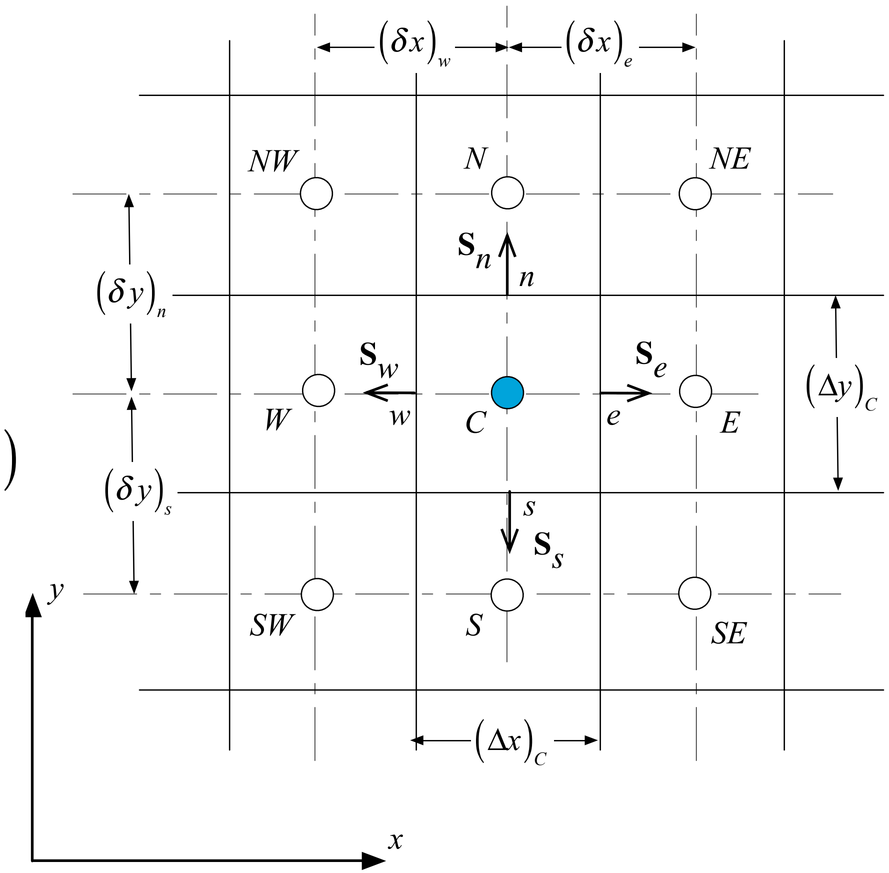
Deferred Correction

$$\sum_{f}\left(\rho\mathbf{v}\phi_{SOU}\right)_f\cdot\mathbf{S}_f=0$$

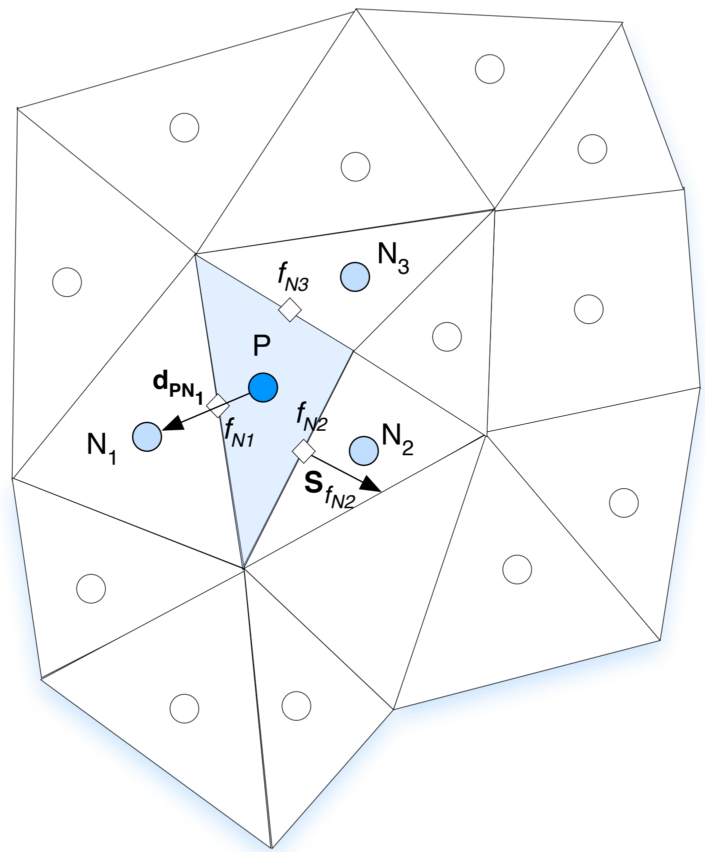
$$\sum_{f=nb(P)}\left(\rho_f\mathbf{v_f}\cdot\mathbf{S_f}\right)\left(\phi_{f,SOU}+\phi_{f,UPWIND}-\phi_{f,UPWIND}\right)=0$$

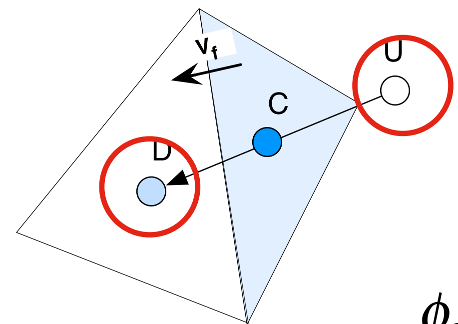
$$\sum_{f=nb(P)}\left(\rho_f\mathbf{v_f}\cdot\mathbf{S_f}\right)\phi_{f,UPWIND}=-\sum_{f=nb(P)}\left(\rho_f\mathbf{v_f}\cdot\mathbf{S_f}\right)\left(\phi_{f,SOU}-\phi_{f,UPWIND}\right)$$

$$a_P\phi_P+\sum_{N=NB(P)}a_N\phi_N=b_P^{dc}$$

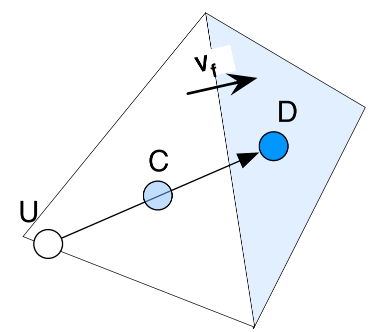


Unstructured Grids





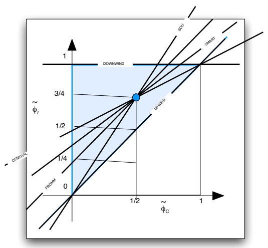
$$\phi_D - \phi_U = \nabla \phi_C \cdot 2\mathbf{r}_{\mathbf{CD}}$$



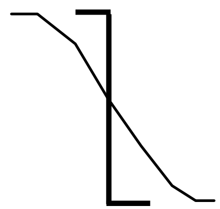
Unstructured Grid

Convection Scheme	Formula
UPWIND	no change
CENTRAL	$\phi_f = \phi_C + \nabla \phi_f \cdot \mathbf{r_{Cf}}$
SOU (Second Order Upwind)	$\phi_f = \phi_C + \left(2\nabla \phi_C - \nabla \phi_f\right) \cdot \mathbf{r_{Cf}}$
QUICK (Quadratic Upwind Interpolation	$\phi_f = \phi_C + \frac{1}{2}\left(\nabla \phi_f + \nabla \phi_C\right) \cdot \mathbf{r_{Cf}}$
DOWNWIND	no change

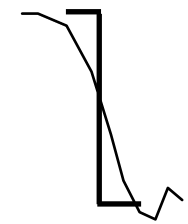
Numerical Dispersion

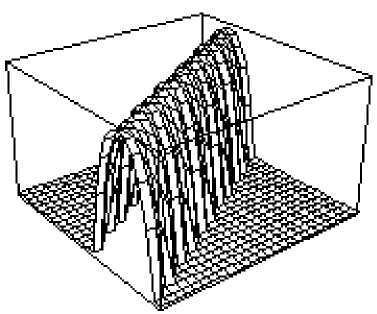


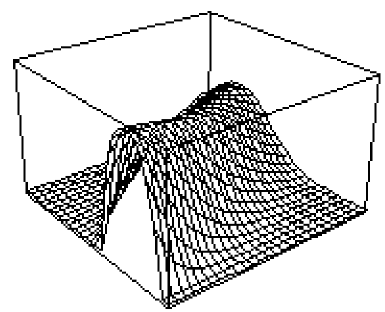
Numerical Diffusion



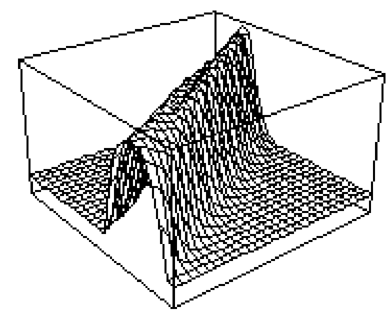
Numerical Dispersion



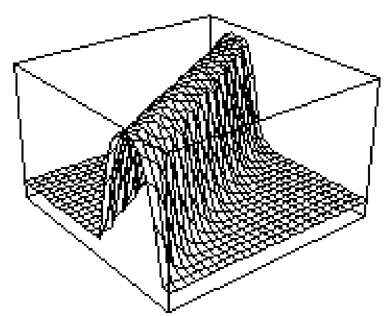


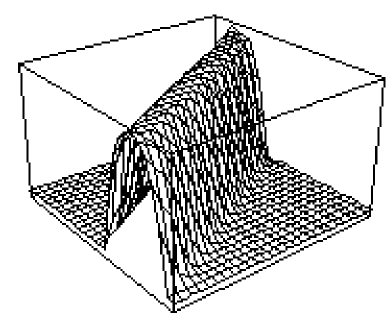


UPWIND



SUDS





STOIC

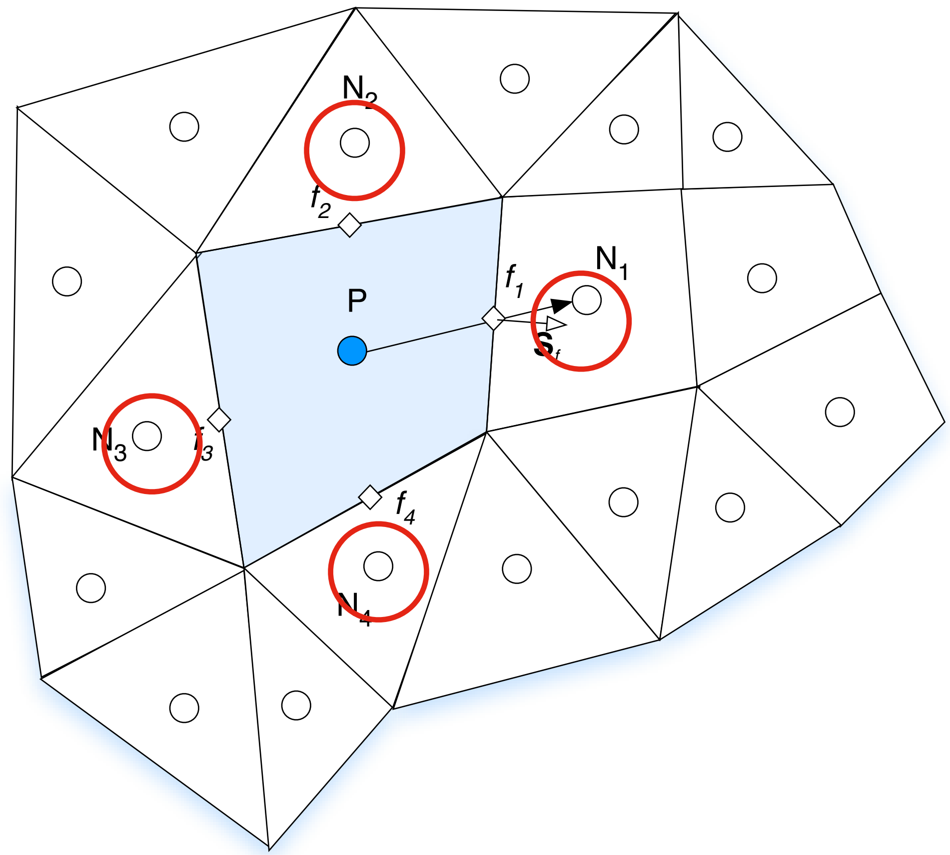
Computing the Gradient

$$\int_V \nabla\phi \, dV = \oint_{\partial V} \phi \, d\mathbf{S}$$

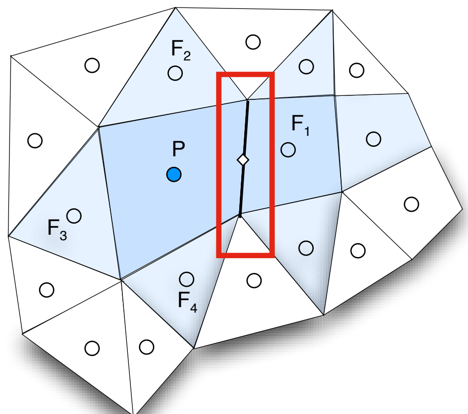
$$\overline{\nabla\phi}\, V = \int_V \nabla\phi \, dV$$

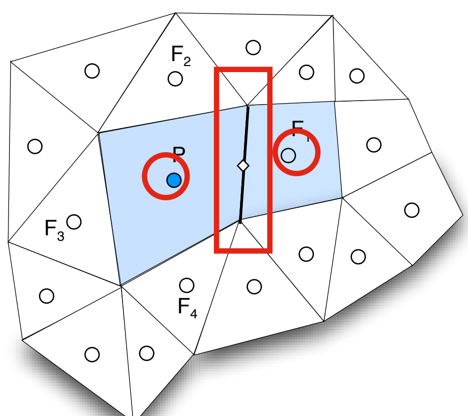
$$\overline{\nabla\phi_P} = \frac{1}{V_P} \oint_{\partial V_P} \phi_f \mathbf{S}_f$$

$$\overline{\nabla\phi}_P = \frac{1}{V_P} \sum_{f=nb(P)} \phi_f \mathbf{S}_f$$



Gradient at the Cell Face





$$\nabla\phi_f = \overline{\nabla\phi_f} - \left(\overline{\nabla\phi_f}\cdot\mathbf{e_{dc}}\right)\mathbf{e_{dc}} + \frac{\phi_D - \phi_C}{\left\|d_{CD}\right\|}\mathbf{e_{dc}}$$

Gradient Formulation of HO schemes

QUICK

$$
\begin{aligned}
\phi_f &= \frac{\phi_C + \phi_D}{2} - \frac{\phi_D - 2\phi_C + \phi_U}{8} \\
&= \phi_C + \frac{\phi_D - \phi_C}{2} - \frac{\phi_D - \phi_C + \phi_U - \phi_C}{8} \\
&= \phi_C + \frac{\phi_D - \phi_C}{2} - \frac{\phi_D - \phi_C + \phi_U - \phi_D + \phi_D - \phi_C}{8} \\
&= \phi_C + \frac{\phi_D - \phi_C}{4} - \frac{\phi_D - \phi_U}{8} \\
&= \phi_C + \frac{1}{2}\left(\nabla\phi_f + \nabla\phi_C\right)\cdot \mathbf{r_{Cf}}
\end{aligned}
$$

SOU

$$
\begin{aligned}
\phi_f &= \frac{3}{2}\phi_C - \frac{1}{2}\phi_U \\
&= \phi_C + \frac{\phi_C - \phi_U}{2} \\
&= \phi_C + \frac{\phi_D - \phi_U}{2} - \frac{\phi_D - \phi_C}{2} \\
&= \phi_C + \left(2\nabla\phi_C - \nabla\phi_f\right)\cdot \mathbf{r_{Cf}}
\end{aligned}
$$

CENTRAL

$$
\begin{aligned}
\phi_f &= \frac{\phi_C + \phi_D}{2} \\
&= \phi_C + \frac{\phi_D - \phi_C}{2} \\
&= \phi_C + \nabla\phi_f \cdot \mathbf{r_{Cf}}
\end{aligned}
$$

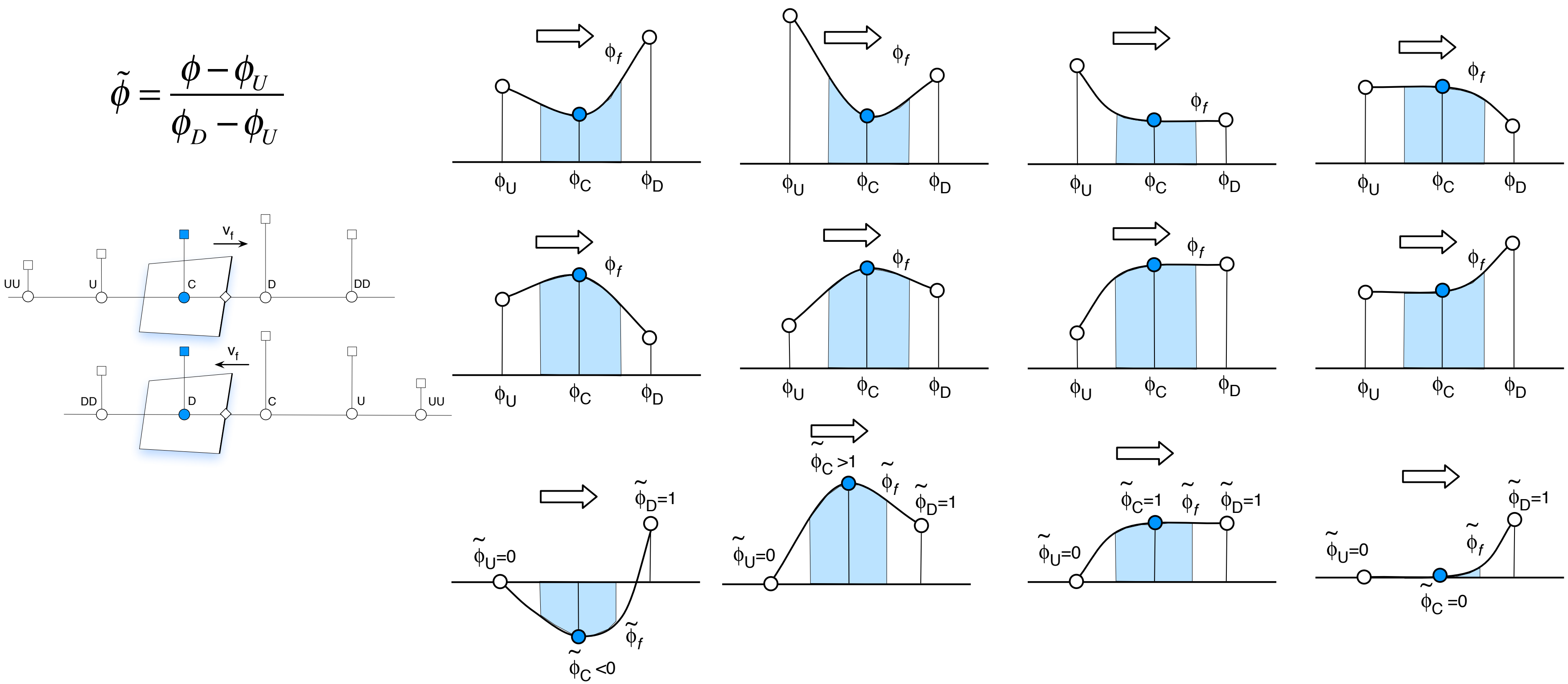
Gradient Form of Schemes

Convection Scheme	Formula
UPWIND	no change
CENTRAL	$\phi_f = \phi_C + \nabla\phi_f \cdot \mathbf{r}_{\mathbf{Cf}}$
SOU (Second Order Upwind)	$\phi_f = \phi_C + \left(2\nabla\phi_C - \nabla\phi_f\right) \cdot \mathbf{r}_{\mathbf{Cf}}$
QUICK (Quadratic Upwind Interpolation	$\phi_f = \phi_C + \frac{1}{2}\left(\nabla\phi_f + \nabla\phi_C\right) \cdot \mathbf{r}_{\mathbf{Cf}}$
DOWNWIND	no change

High Resolution Schemes

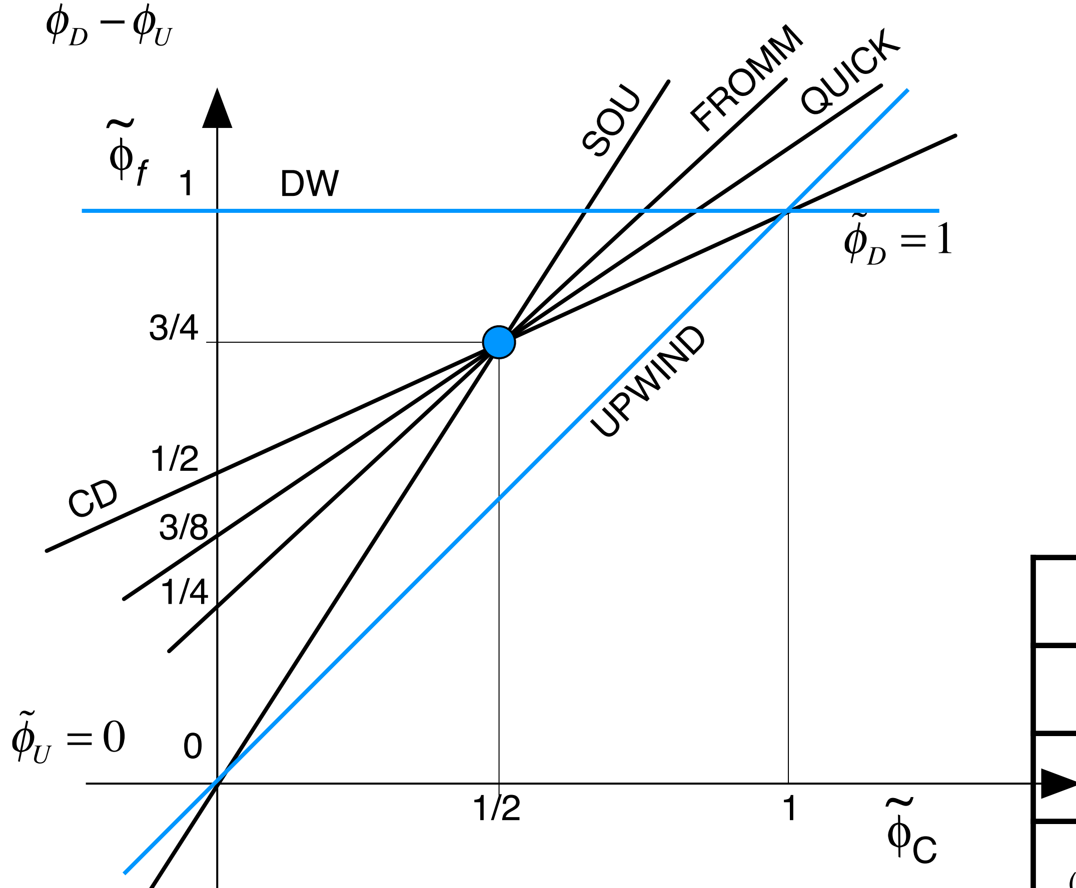
I-Normalization Procedure

$$\tilde{\phi} = \frac{\phi - \phi_U}{\phi_D - \phi_U}$$



Normalized HO Schemes

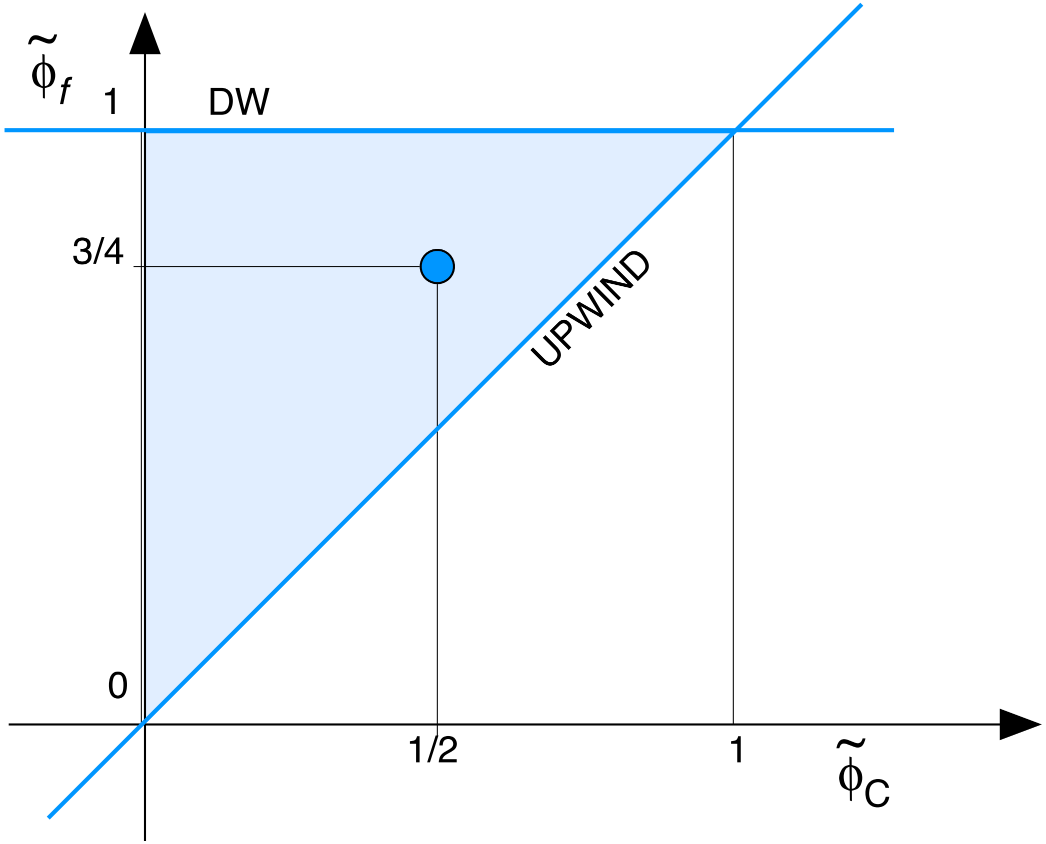
$$\tilde{\phi}_f = \frac{\phi_f - \phi_U}{\phi_D - \phi_U}$$

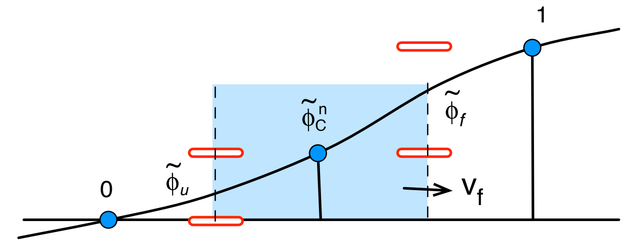


$$\tilde{\phi}_C = \frac{\phi_C - \phi_U}{\phi_D - \phi_U}$$

Scheme	Formula	Normalized
UPWIND	$\phi_f = \phi_C$	$\tilde{\phi}_f = \tilde{\phi}_C$
CENTRAL	$\phi_f = \frac{\phi_C + \phi_D}{2}$	$\tilde{\phi}_f = \frac{1}{2}\tilde{\phi}_C + \frac{1}{2}$
SOU (Second Order Upwind)	$\phi_f = \frac{3}{2}\phi_C - \frac{1}{2}\phi_U$	$\tilde{\phi}_f = \frac{3}{2}\tilde{\phi}_C$
QUICK (Quadratic Upwind Interpolation for	$\phi_f = \frac{3}{4}\phi_C + \frac{1}{8}\phi_D - \frac{3}{8}\phi_D$	$\tilde{\phi}_f = \frac{3}{4}\tilde{\phi}_C + \frac{1}{8}$
DOWNWIND	$\phi_f = \phi_D$	$\tilde{\phi}_f = 1$

2-CBC Criteria





$$
\tilde{\phi}_f = \begin{cases} f\left(\tilde{\phi}_C\right) \quad continuous \\ f\left(\tilde{\phi}_C\right) = 1 \quad if\ \tilde{\phi}_C = 1 \\ \tilde{\phi}_C < f\left(\tilde{\phi}_C\right) < 1 \quad 0 < \tilde{\phi}_C < 1 \\ f\left(\tilde{\phi}_C\right) = 0 \quad if\ \tilde{\phi}_C = 0 \\ f\left(\tilde{\phi}_C\right) = \tilde{\phi}_C \quad if\ \tilde{\phi}_C < 0\ or\ \tilde{\phi}_C > 1 \end{cases}
$$

$$\min\left(\phi_{NB},\phi_C\right) \le \phi_f \le \max\left(\phi_{NB},\phi_C\right)$$

Boundedness Criteria

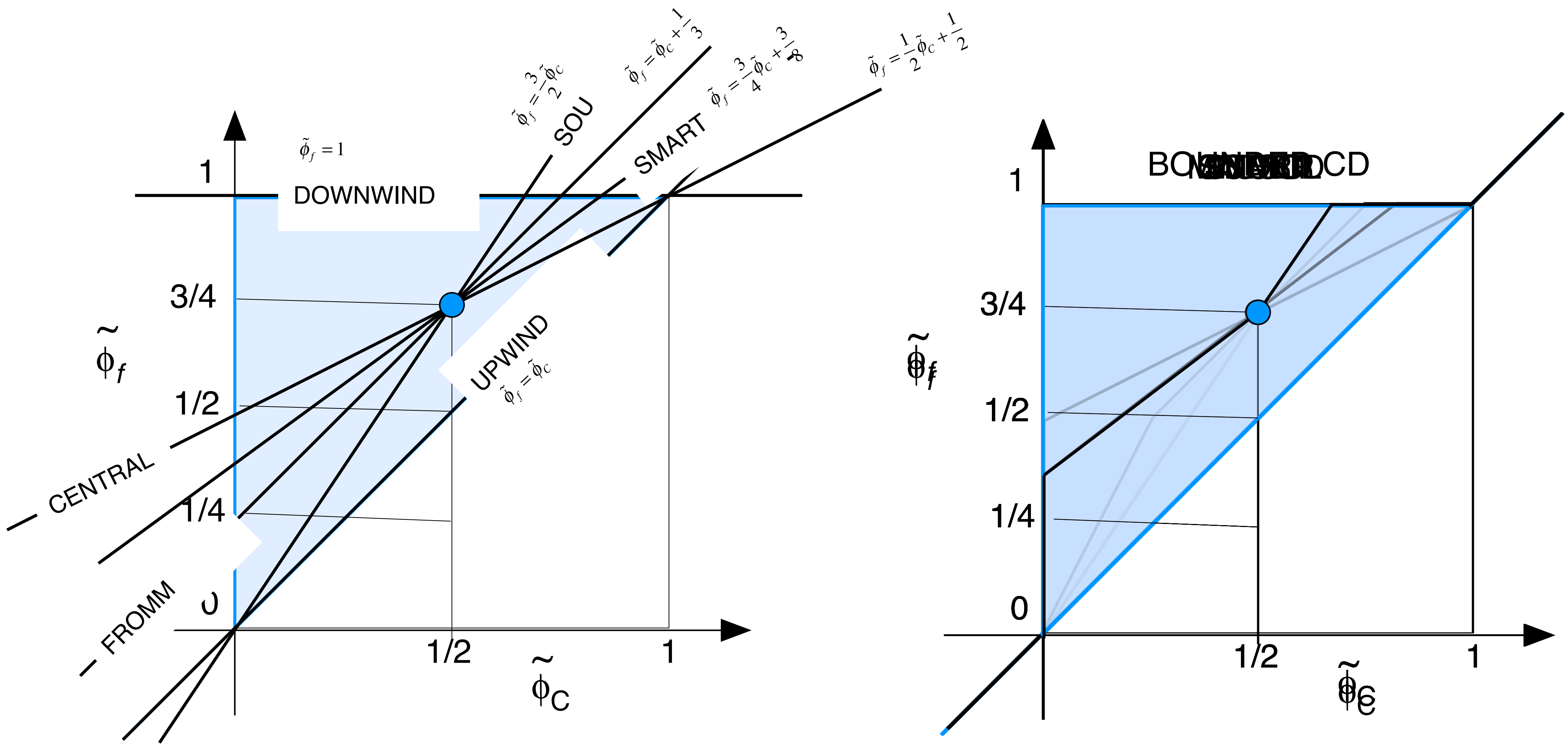
$$\phi_i^{n+1} = H\left(\phi_{i-k}^n,\phi_{i-k+1}^n,\ldots,\phi_{i+k}^n\right)$$

Flux Limiter (TVD)

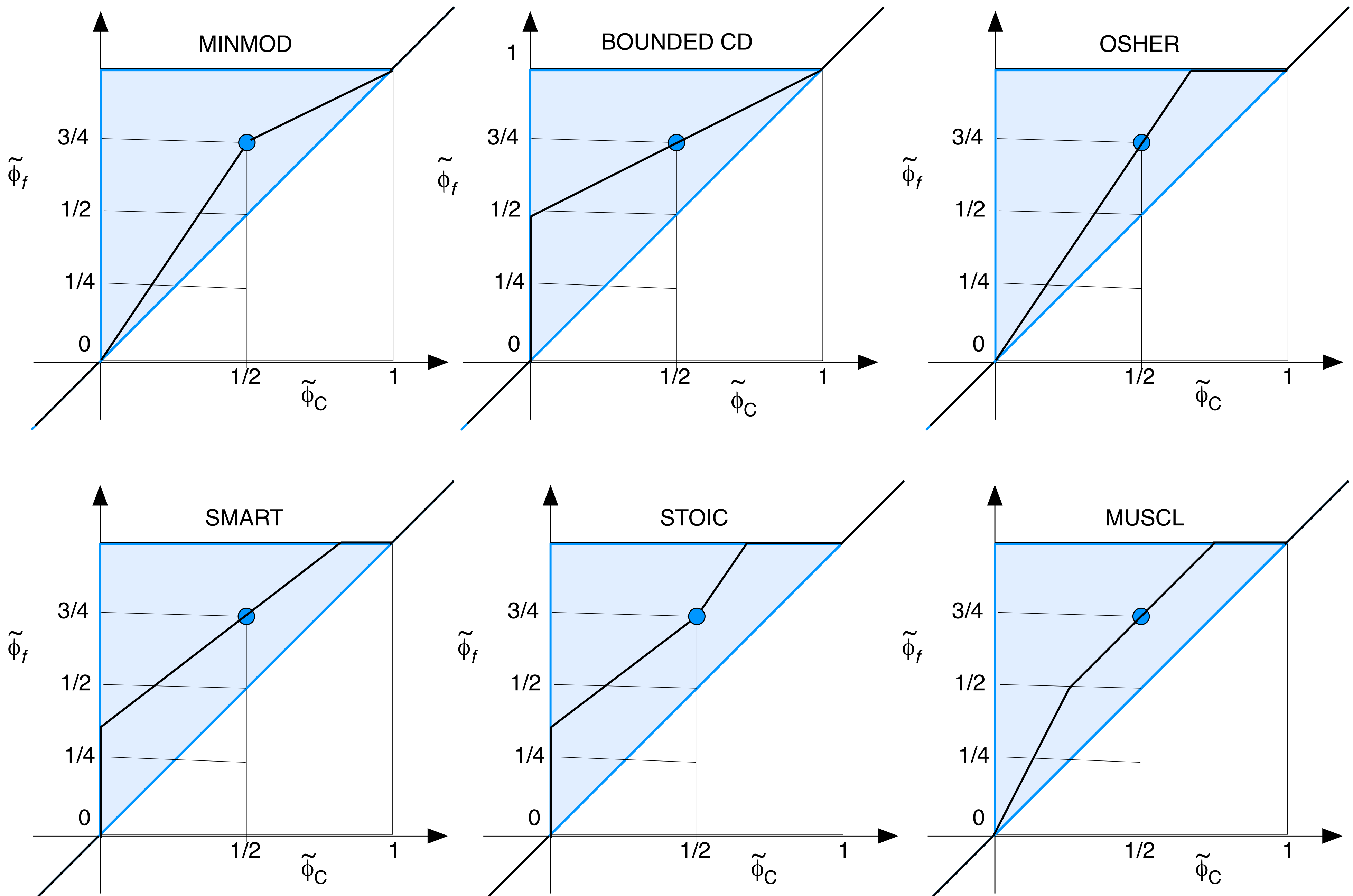
$$\frac{\partial H}{\partial \phi_j^n} \ge 0 \quad \forall j \in [i-k, i+k]$$

Positiveness

The Normalized Variable Diagram

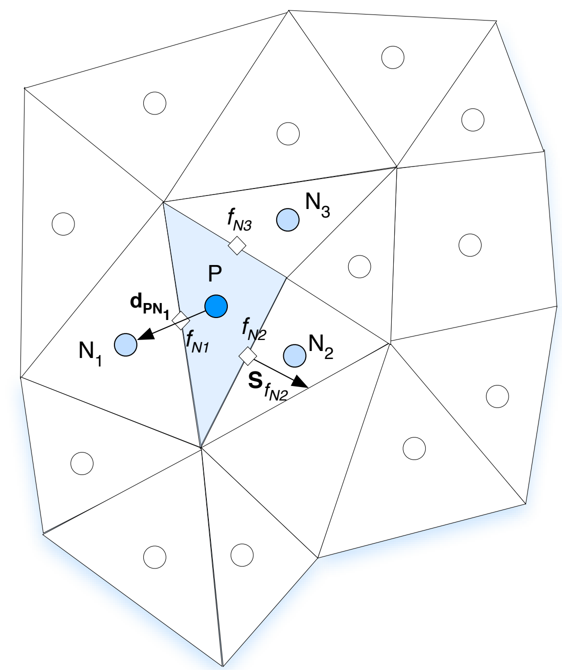


Bounded HR Schemes



Implementation

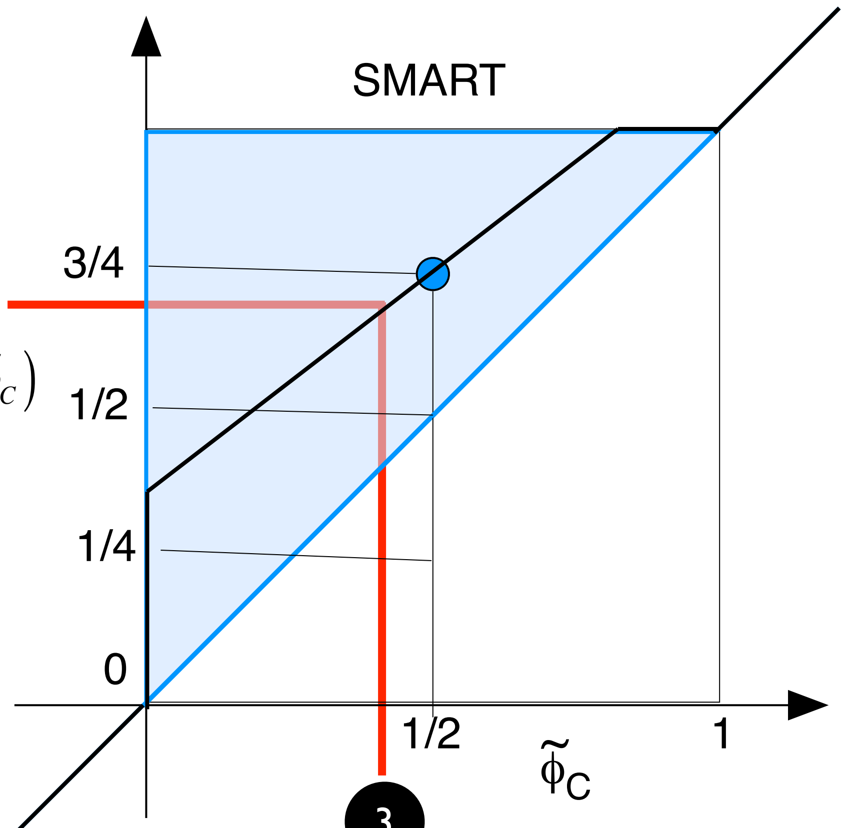
1 $\phi_P, \phi_{N_1}, \phi_{N_2}, \phi_{N_3}$



2 $\phi_C \leftarrow \phi_P$

$\phi_D \leftarrow \phi_{N_1}$

$\phi_U \leftarrow \phi_D - \nabla \phi_P \cdot 2\mathbf{r}_{PN_1}$



3 $\tilde{\phi}_C = \dfrac{\phi_C - \phi_U}{\phi_D - \phi_U}$

4 $\tilde{\phi}_f = f\left(\tilde{\phi}_C\right)$

5 $\phi_f \leftarrow f\left(\tilde{\phi}_C\right)\left(\phi_D - \phi_U\right) + \phi_U$

Deferred Correction

$$\sum_{f=nb(P)} \left(\rho_f \mathbf{v}_f \phi_f\right) \cdot \mathbf{S}_f = \sum_{f=nb(P)} \dot{m}_f \phi_f = 0$$

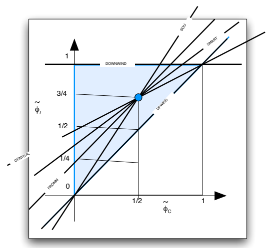
$$\begin{aligned}
\rho_f \mathbf{v}_f \phi_f^{SMART} \cdot \mathbf{S}_f &= \left(\rho_f \mathbf{v}_f \cdot \mathbf{S}_f\right) \phi_f^{SMART} \\
&= \dot{m}_f \phi_f^{SMART} \\
&= \underbrace{\dot{m}_f \phi_f^{UPWIND}}_{\text{treat implicitly}} + \underbrace{\dot{m}_f \left(\phi_f^{SMART} - \phi_f^{UPWIND}\right)}_{\text{treat explicitly}}
\end{aligned}$$

$$\sum_{f=nb(P)} \dot{m}_f \phi_f^{SMART} = \sum_{f=nb(P)} \underbrace{\dot{m}_f \phi_f^{UPWIND}}_{implicit} + \sum_{f=nb(P)} \underbrace{\dot{m}_f \left(\phi_f^{SMART} - \phi_f^{UPWIND}\right)}_{deferred}$$

$$a_P \phi_P^{UPWIND} + \sum_{F=NB(P)} a_F \phi_F^{UPWIND} = b_P - \sum_{f=nb(P)} \underbrace{\dot{m}_f \left(\phi_f^{SMART} - \phi_f^{UPWIND}\right)}_{deferred}$$

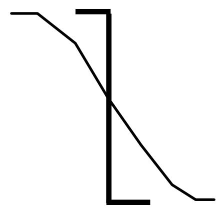
$$a_P \phi_P^{UPWIND} + \sum_{F=NB(P)} a_F \phi_F^{UPWIND} = b_P + b_P^{dc-SMART}$$

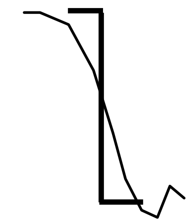
Numerical Dispersion

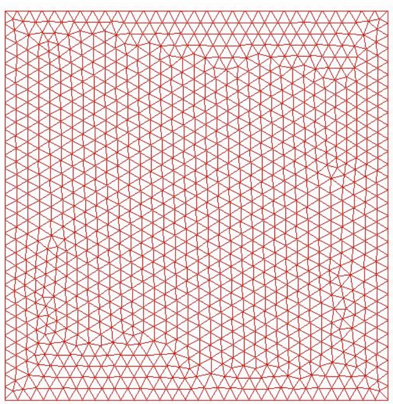


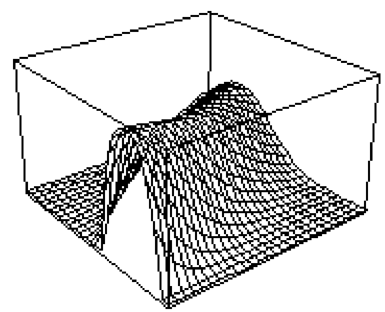
Numerical Diffusion

Numerical Dispersion

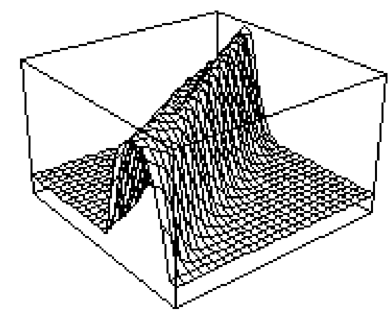




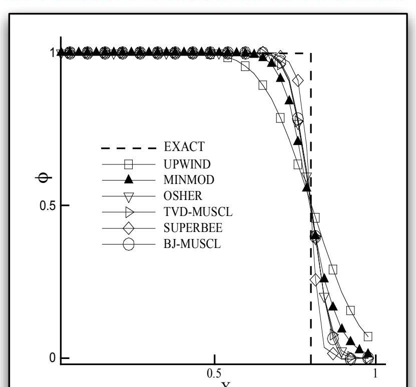


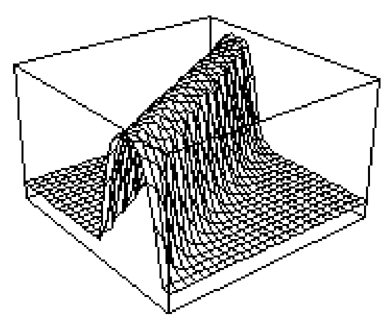


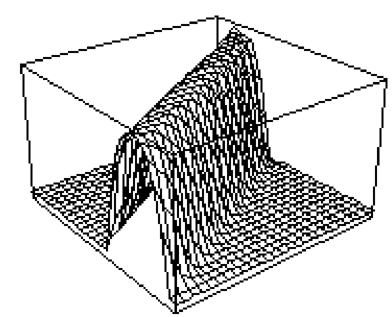
UPWIND



SUDS

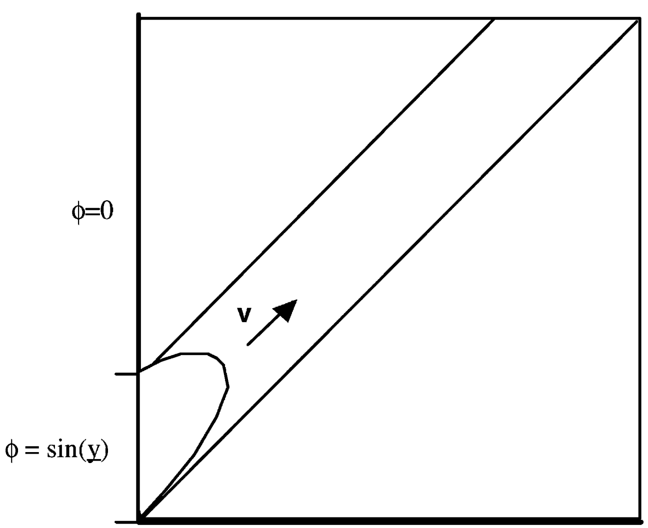




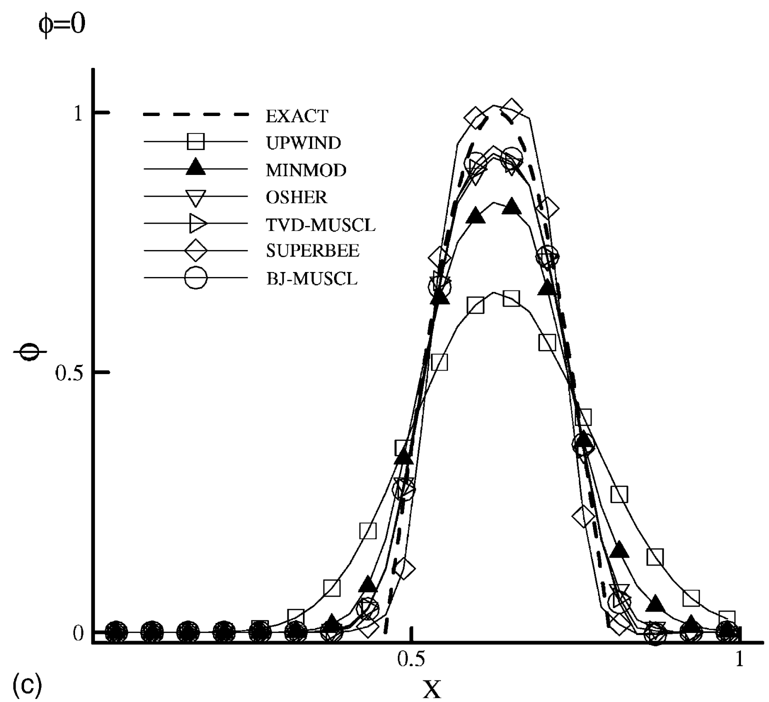


STOIC

HR Schemes



(a)



(c)