Discretization of the Convection Term

Chapter 11

Convection Term

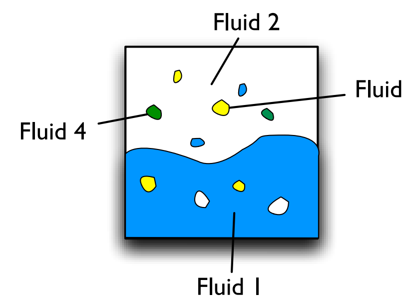
$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}\mathbf{v})}=\nabla\cdot\tau+\mathbf{B}$$

$$\frac{\partial(\rho E)}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}E)}=-\nabla\cdot(\mathbf{v}P)+\nabla\cdot(\mathbf{v}\cdot\tau)+Q$$

$$\frac{\partial(\rho\alpha)}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}\alpha)}=0$$

$$\frac{\partial(\rho\alpha C)}{\partial t}+\underline{\nabla\cdot(\rho\mathbf{v}\alpha C)}=0$$

$$\boxed{\frac{\partial(\rho\phi)}{\partial t}+\nabla\cdot(\rho\mathbf{v}\phi)=\nabla\cdot(\Gamma\nabla\phi)+Q}$$

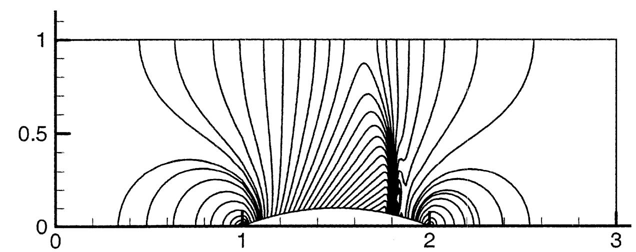


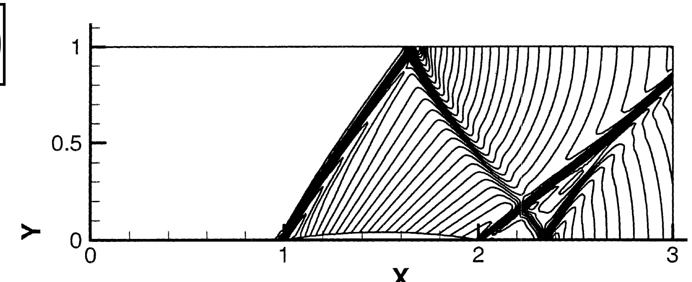
Pressure Correction Equation

Transient-like Term

$$\boxed{\frac{\Omega C_\rho}{\Delta t}(P'_P)}+\boxed{\sum_f\left(C_\rho U_f^* P'\right)_f}-\sum_f\left(\rho_f^*\overline{\mathrm{D}}_f(\nabla P')_f\cdot\mathbf{S}_f\right)=-\boxed{\frac{\Omega}{\Delta t}\left(\rho_P^*-\rho_P^o\right)}+\sum_f\left(\rho_f^* U_f^*\right)-\sum_f\left(\rho'_f\mathbf{v}'_f\cdot\mathbf{S}_f\right)-\sum_f\left(\rho_f^*\overline{\mathrm{H}[\mathbf{v}']}_f\cdot\mathbf{S}_f\right)$$

Advection-like Term
account for compressibility effects

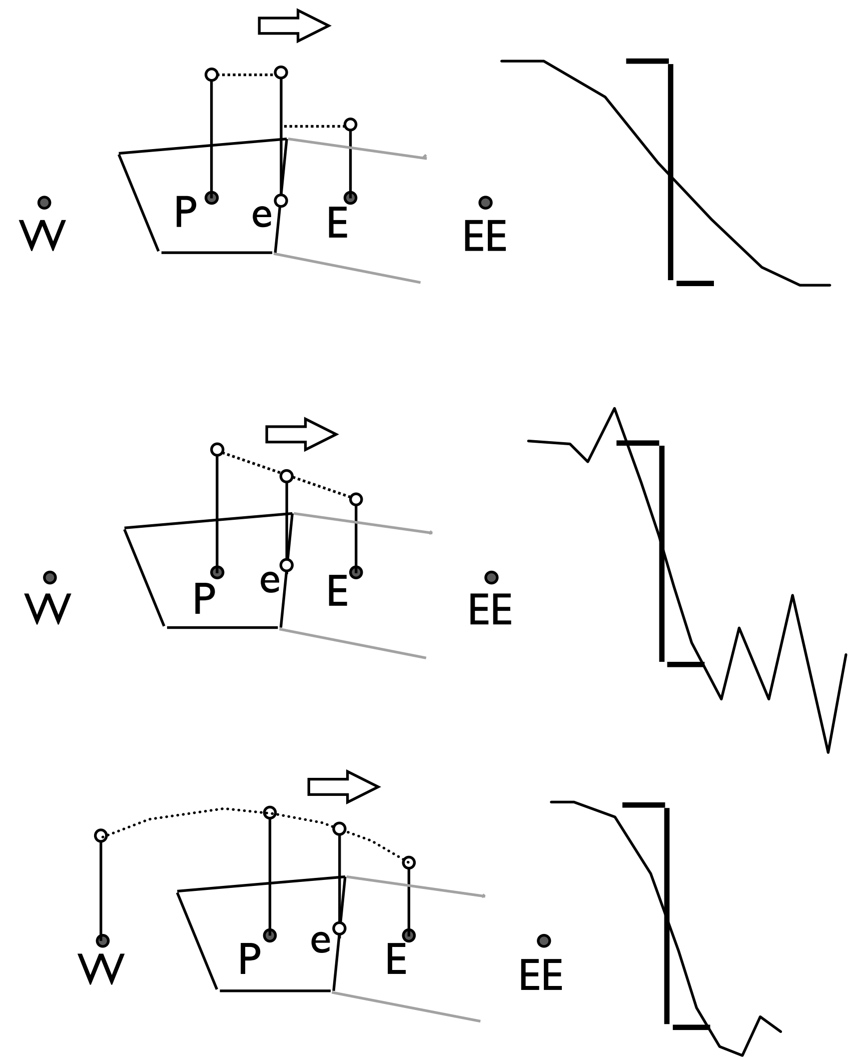




The simplicity of the convection term does not extend to its discretization (3 decades of research and continuing)

Source of Errors

- Numerical Diffusion
  - Smearing of the predicted profile
  - Stream-wise Numerical Diffusion
    - low order interpolation profile
  - Cross-stream Numerical Diffusion
    - one-dimensional nature of the assumed profile
- Numerical Dispersion
  - Non-physical spatial oscillations or over/under shoots
  - Unphysical behavior of the assumed profile

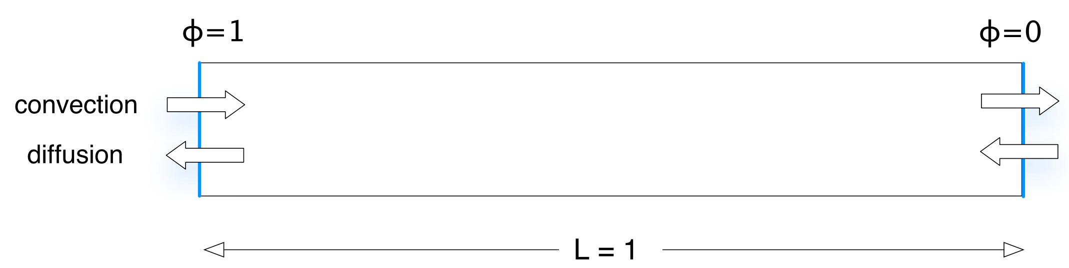


Convection-Diffusion Equation

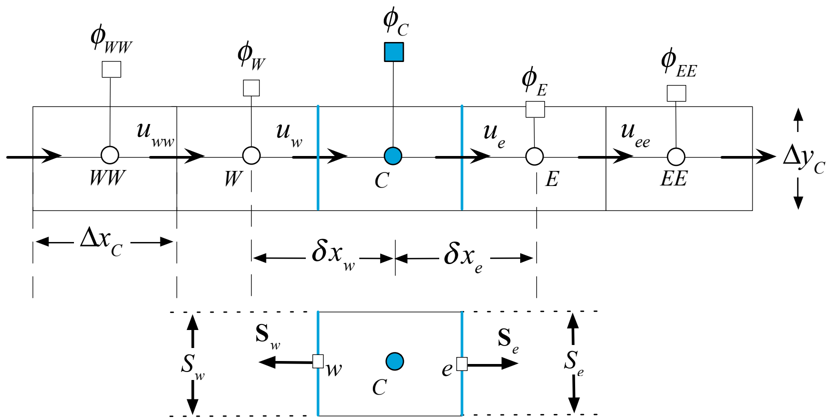
$$\nabla \cdot (\rho \mathbf{v} \phi) - \nabla \cdot (\Gamma \nabla \phi) = 0$$

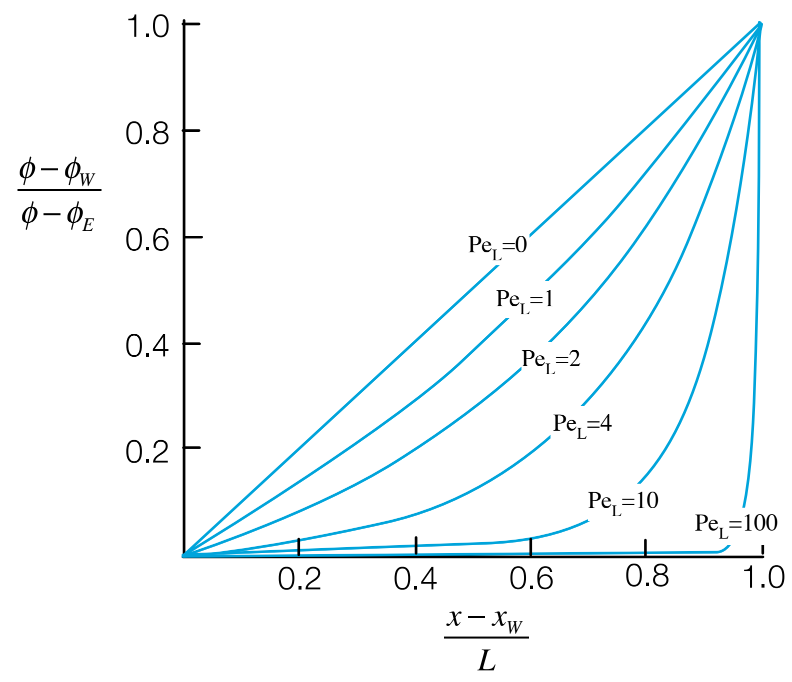
- The simplicity of the convection term does not extend to its discretization (3 decades of research and more is still needed)

$$\frac{d(\rho u \phi)}{dx} - \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) = 0$$



Nomenclature

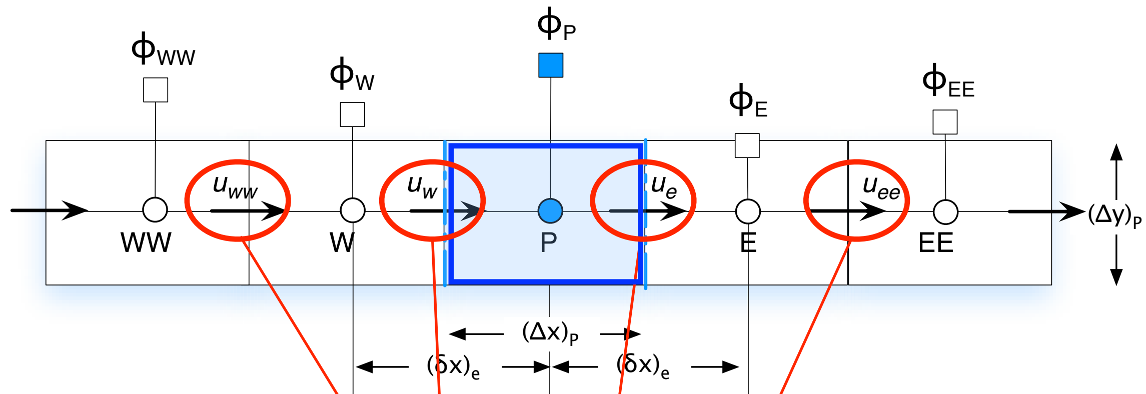




1D Cartesian

$$\sum_{\sim f=nb(P)} (\rho\mathbf{v}\phi - \Gamma\nabla\phi)_f \cdot \mathbf{S}_f = 0$$

$u > 0$



$$\mathbf{v} = u\mathbf{i} \qquad \nabla\phi = \frac{d\phi}{dx}\mathbf{i}$$

$$\mathbf{S}_e = \Delta y\mathbf{i}$$

$$\mathbf{S}_w = -\Delta y\mathbf{i}$$

$$\sum_{\sim f=nb(P)} \left(\rho u\mathbf{i}\phi - \Gamma\frac{d\phi}{dx}\mathbf{i}\right)_f \cdot \mathbf{S}_f = 0$$

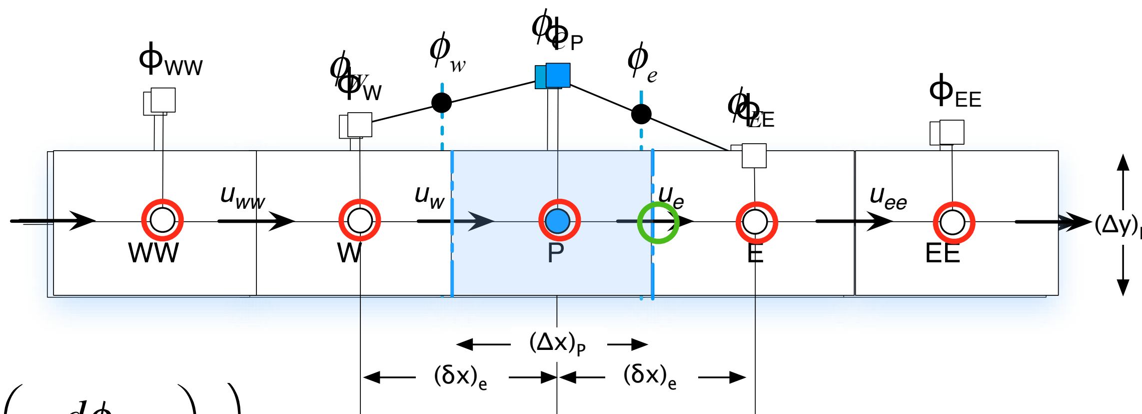
$$(\rho u\Delta y\phi)_e - (\rho u\Delta y\phi)_w - \left(\left(\Gamma\frac{d\phi}{dx}\Delta y\right)_e - \left(\Gamma\frac{d\phi}{dx}\Delta y\right)_w\right) = 0$$

Numerical Instability

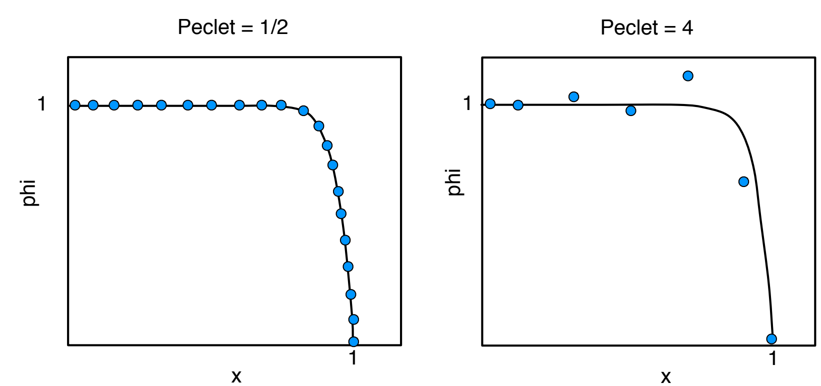
$$u > 0$$

$$(\rho u \Delta y)_e \phi_e \qquad \phi_e = \frac{(\phi_E + \phi_P)}{2}$$

$$(\rho u \Delta y)_w \phi_w \qquad \phi_w = \frac{(\phi_W + \phi_P)}{2}$$



$$(\rho u \Delta y \phi)_e - (\rho u \Delta y \phi)_w - \left(\left(\Gamma \frac{d\phi}{dx} \Delta y \right)_e - \left(\Gamma \frac{d\phi}{dx} \Delta y \right)_w \right) = 0$$



$$(\rho u \Delta y)_e \frac{(\phi_E + \phi_P)}{2} - (\rho u \Delta y)_w \frac{(\phi_P + \phi_W)}{2}$$

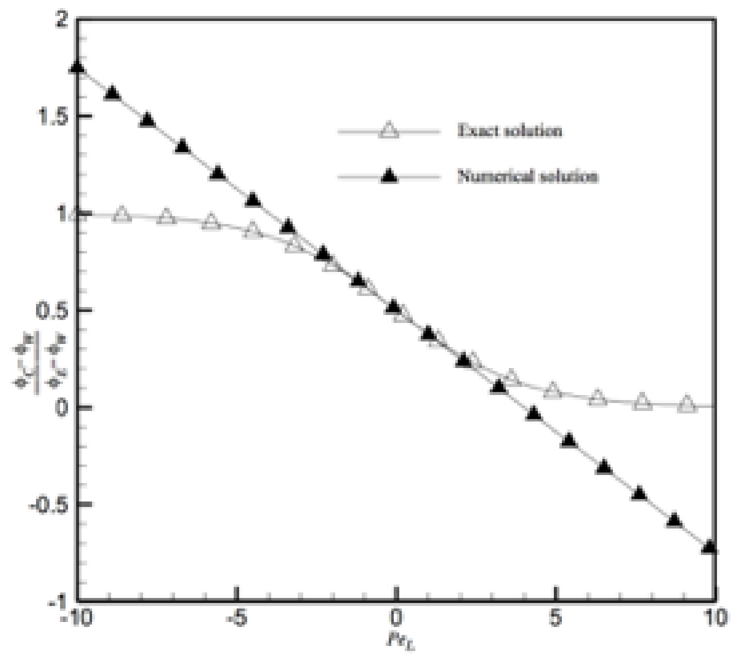
$$-\left(\left(\Gamma \frac{\Delta y}{\delta x} \right)_e (\phi_E - \phi_P) - \left(\Gamma \frac{\Delta y}{\delta x} \right)_w (\phi_P - \phi_W) \right) = 0$$

$$a_P \phi_P + a_E \phi_E + a_W \phi_W = 0$$

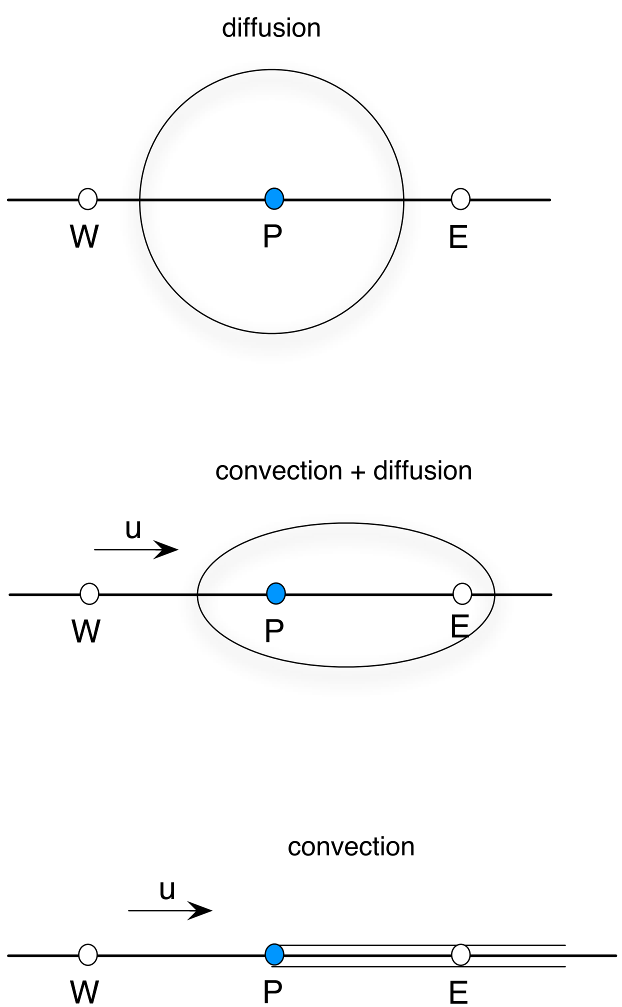
$$a_P^{Conv} = \left(\frac{\rho u \Delta y}{2} \right)_e - \left(\frac{\rho u \Delta y}{2} \right)_w$$

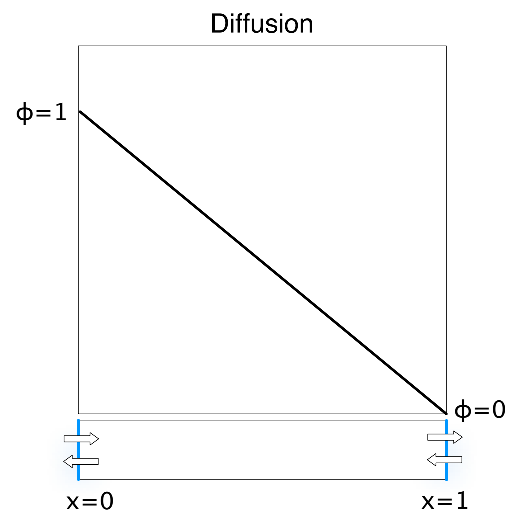
if it gets ≤ 0 we have numerical Instabilities !

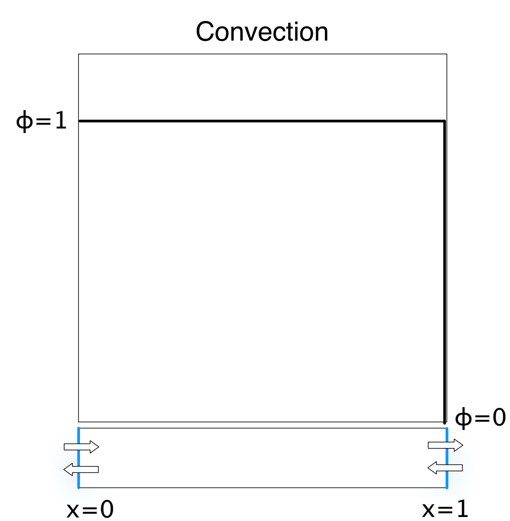
Analytical vs Numerical Solution

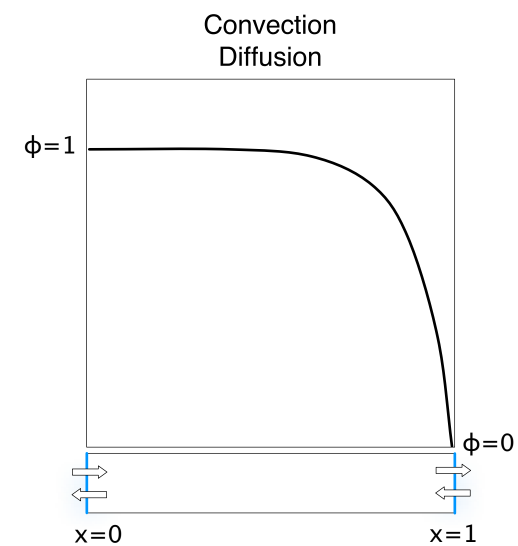


Term Influences









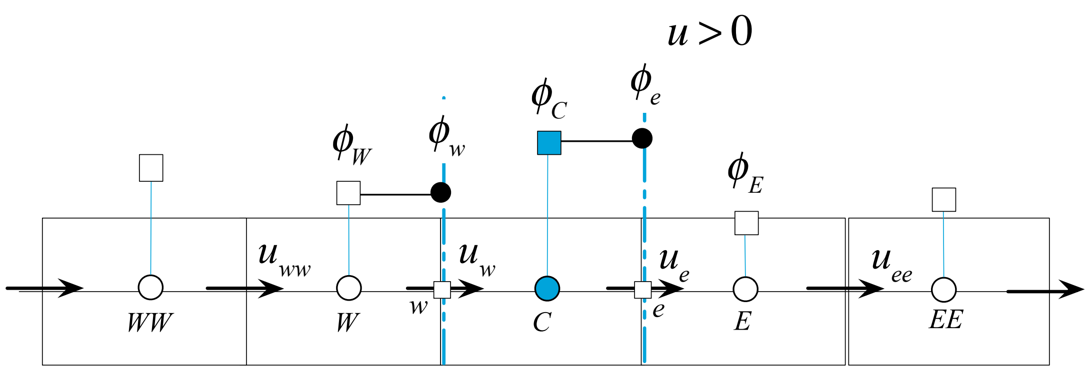
Upwind Scheme

$$\phi_e = k_0 = \begin{cases} \phi_P \ \text{if} \ u_e > 0 \\ \phi_E \ \text{if} \ u_e < 0 \end{cases}$$

$$\phi_w = k_{0w} = \begin{cases} \phi_W \ \text{if} \ u_w > 0 \\ \phi_P \ \text{if} \ u_w < 0 \end{cases}$$

$$(\rho u \Delta y)_e \phi_P - (\rho u \Delta y)_w \phi_W$$

$$-\left(\Gamma \frac{S}{\delta x}\right)_e (\phi_E - \phi_P) - \left(\Gamma \frac{S}{\delta x}\right)_w (\phi_W - \phi_P) = 0$$



$$a_P \phi_P + a_E \phi_E + a_W \phi_W = 0$$

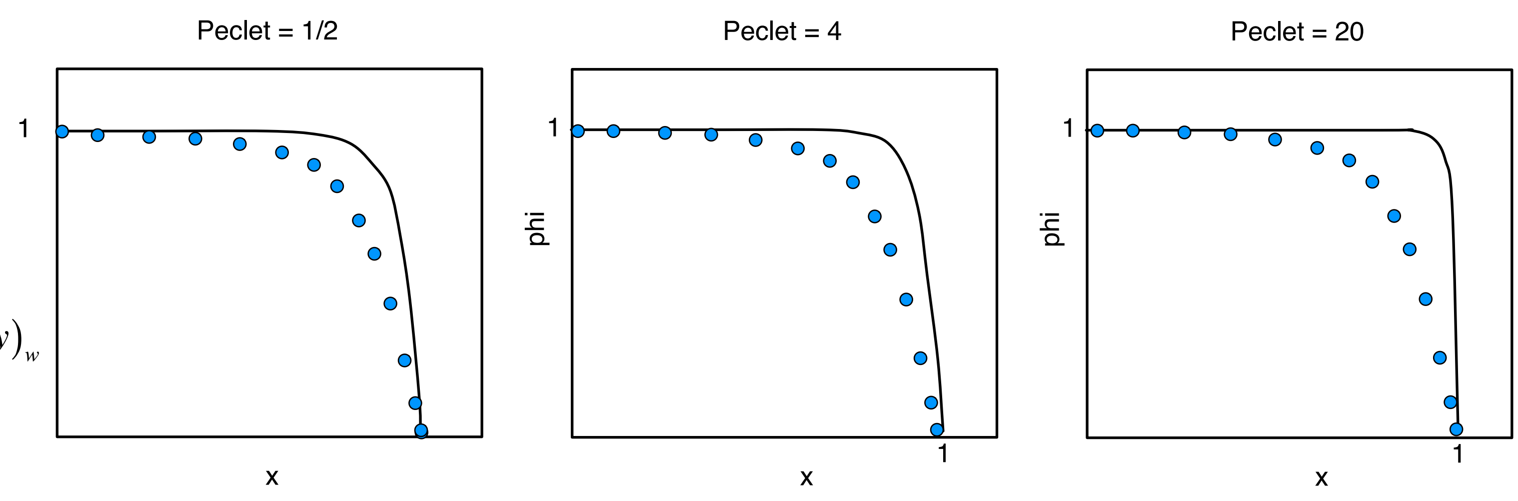
$$(\rho u \Delta y \phi)_e - (\rho u \Delta y \phi)_w - \left(\left(\Gamma \frac{d\phi}{dx} \Delta y\right)_e - \left(\Gamma \frac{d\phi}{dx} \Delta y\right)_w\right) = 0$$

$$a_E = -\Gamma \frac{S_e}{\delta x_e}$$

$$a_W = -\Gamma \frac{S_w}{\delta x_w} - (\rho u \Delta y)_w$$

$$a_P = \Gamma \frac{S_e}{\delta x_e} + \Gamma \frac{S_w}{\delta x_w} + (\rho u \Delta y)_e$$

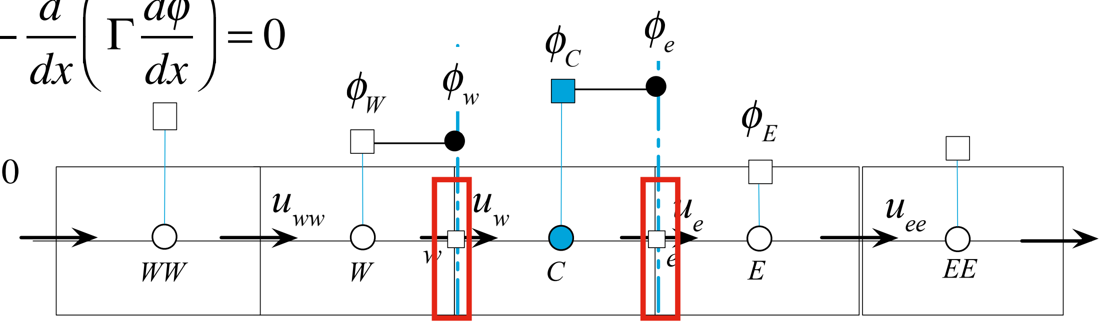
$$= -(a_E + a_W) + (\rho u \Delta y)_e - (\rho u \Delta y)_w$$



Numerical Diffusion

Upwind Scheme

$$\frac{d(\rho u \phi)}{dx} - \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) = 0$$



$$(\rho u \Delta y)_e \phi_e - (\rho u \Delta y)_w \phi_w - \left(\Gamma \frac{d\phi}{dx} \Delta y\right)_e + \left(\Gamma \frac{d\phi}{dx} \Delta y\right)_w = 0$$

$$(\rho u \Delta y)_e \phi_P - (\rho u \Delta y)_w \phi_W - \left(\Gamma \frac{d\phi}{dx} \Delta y\right)_e + \left(\Gamma \frac{d\phi}{dx} \Delta y\right)_w = 0$$

Taylor expansion

$$\phi_P = \phi_e + \left(\frac{\partial \phi}{\partial x}\right)_e (x_P - x_e) + O(x_P - x_e)^2$$

$$= \phi_e - \left(\frac{\partial \phi}{\partial x}\right)_e \frac{\delta x_e}{2} + O(x_P - x_e)^2$$

$$\phi_W = \phi_w + \left(\frac{\partial \phi}{\partial x}\right)_w (x_W - x_w) + O(x_W - x_w)^2$$

$$= \phi_w - \left(\frac{\partial \phi}{\partial x}\right)_w \frac{\delta x_w}{2} + O(x_W - x_w)^2$$

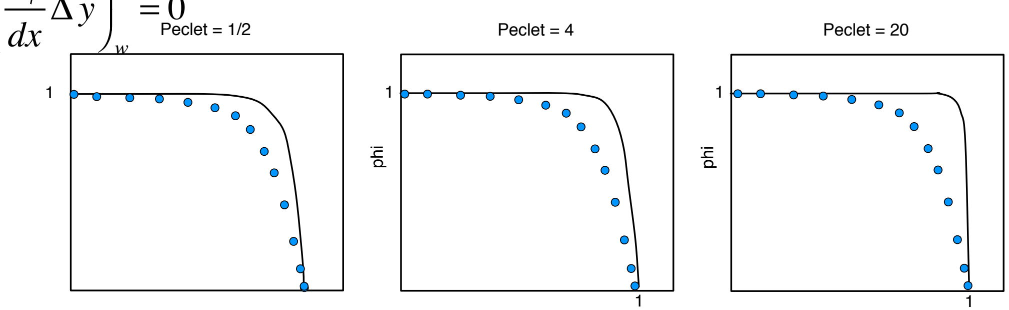
$$(\rho u \Delta y)_e \left\{\phi_e + \left(\frac{\partial \phi}{\partial x}\right)_e \frac{\delta x_e}{2}\right\} - (\rho u \Delta y)_w \left\{\phi_w - \left(\frac{\partial \phi}{\partial x}\right)_w \frac{\delta x_w}{2}\right\}$$

$$-\left(\Gamma \frac{d\phi}{dx} \Delta y\right)_e + \left(\Gamma \frac{d\phi}{dx} \Delta y\right)_w = 0$$

Re-arranging

$$(\rho u \Delta y)_e \phi_e - (\rho u \Delta y)_w \phi_w - \left[\boxed{\frac{\rho u \delta x_e}{2}} + \Gamma_e\right]\left(\frac{d\phi}{dx} \Delta y\right)_e + \left[\boxed{\frac{\rho u \delta x_w}{2}} + \Gamma_w\right]\left(\frac{d\phi}{dx} \Delta y\right)_w = 0$$

$$\Gamma_{truncation} = \rho u \frac{\delta x}{2}$$



Stability Criterion

$$\frac{d(\rho\phi)}{dt} = \boxed{-\frac{d(\rho u\phi)}{dx}} + \boxed{\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)}$$

Central: $$\frac{d(\rho\phi)}{dt} = -\left\{(\rho u\Delta y)_e\frac{(\phi_E+\phi_P)}{2} - (\rho u\Delta y)_w\frac{(\phi_P+\phi_W)}{2}\right\} + \left\{\left(\Gamma\frac{\Delta y}{\delta x}\right)_e(\phi_E-\phi_P) - \left(\Gamma\frac{\Delta y}{\delta x}\right)_w(\phi_P-\phi_W)\right\}$$

Upwind: $$\frac{d(\rho\phi)}{dt} = -\left\{(\rho u\Delta y)_e\phi_P - (\rho u\Delta y)_w\phi_W\right\} + \left\{\left(\Gamma\frac{\Delta y}{\delta x}\right)_e(\phi_E-\phi_P) - \left(\Gamma\frac{\Delta y}{\delta x}\right)_w(\phi_P-\phi_W)\right\}$$

If an error exist at time t, we want errors to grow smaller with time.

This can be achieved if

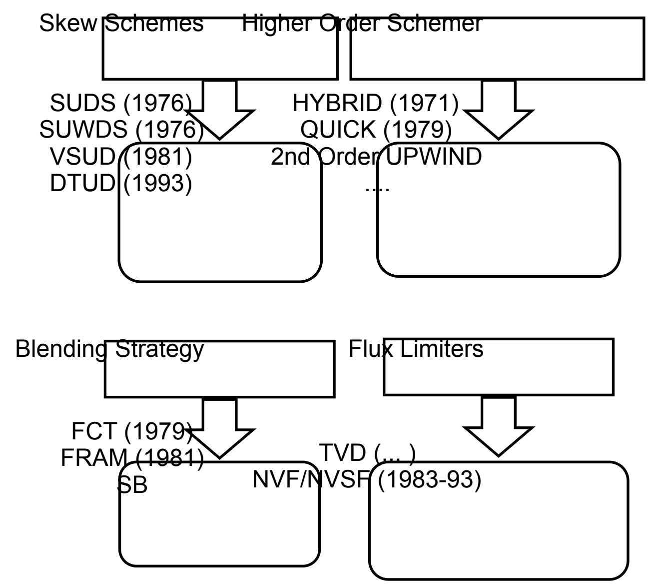
$$\frac{\partial(RHS)}{\partial\phi_P} < 0$$

$$\frac{\partial(RHS_{diffusion})}{\partial\phi_P} = -2\Gamma\frac{\Delta y}{\delta x} < 0$$ always stable

$$\frac{\partial(RHS_{CENTRAL})}{\partial\phi_P} = -\frac{1}{2}\left((\rho u\Delta y)_e - (\rho u\Delta y)_w\right) \approx 0$$ unstable

$$\frac{\partial(RHS_{UPWIND})}{\partial\phi_P} = -(\rho u\Delta y)_e < 0$$ always stable

Strategies



Numerical Dispersion

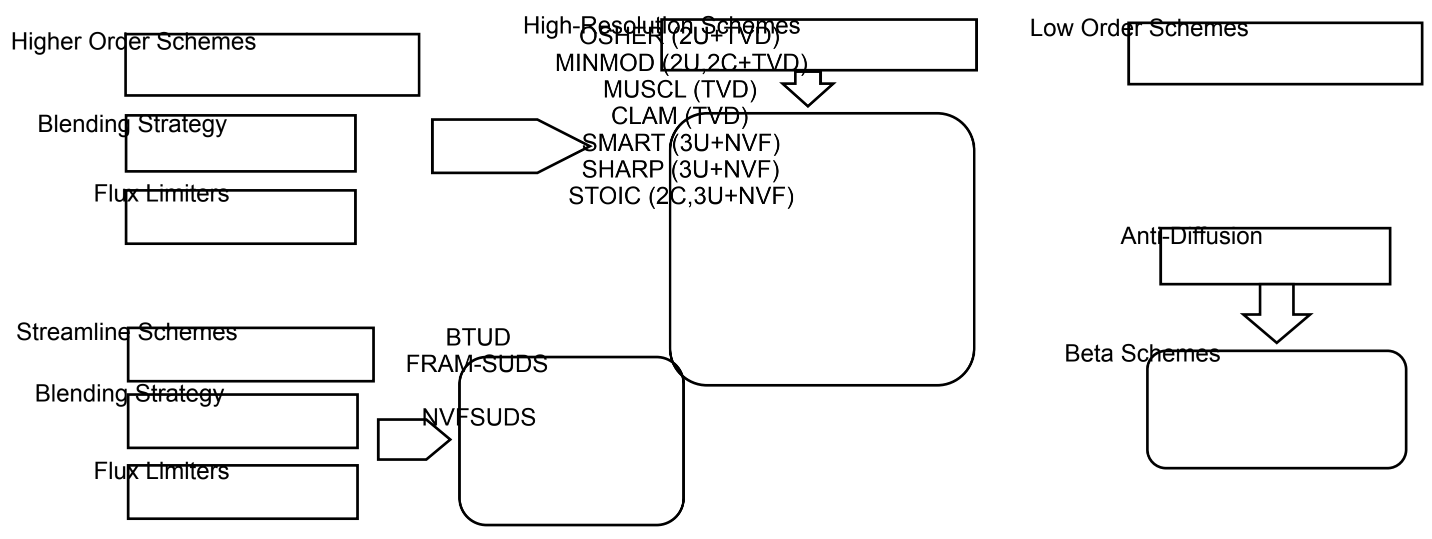
Blending Strategy

A high order scheme is blended with a low order scheme

Composite Flux Limiter

A Limiter function is applied to a flux calculated using a high-order scheme

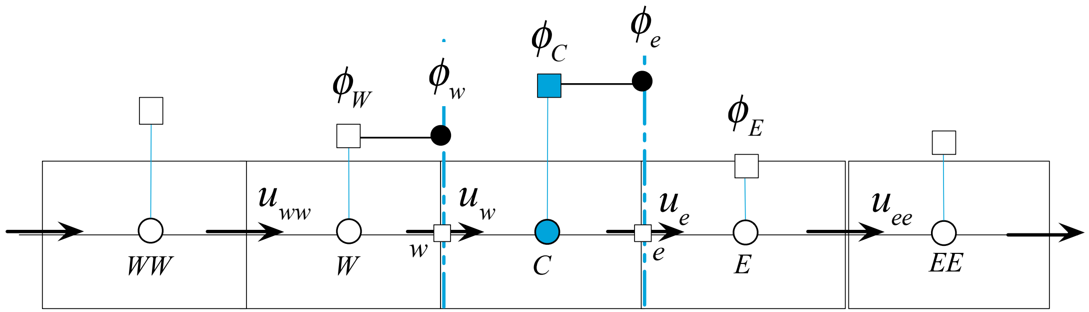
NVF
TVD
Anti-Diffusive Schemes



Hybrid Scheme

The idea is to combine the Central and Upwind Schemes, so as to add robustness of the Upwind scheme to the Central scheme

$$a_P\phi_P + a_E\phi_E + a_W\phi_W = 0$$



$u > 0$

CENTRAL

$$a_E = -\Gamma\frac{\Delta y_e}{\delta x_e} + \frac{(\rho u \Delta y)_e}{2}$$

$$a_W = -\Gamma\frac{\Delta y_w}{\delta x_w} - \frac{(\rho u \Delta y)_w}{2}$$

$$a_P = \Gamma\frac{\Delta y_e}{\delta x_e} + \Gamma\frac{\Delta y_w}{\delta x_w} + \frac{(\rho u \Delta y)_e}{2} - \frac{(\rho u \Delta y)_w}{2}$$
$$= -(a_E + a_W) + (\rho u \Delta y)_e - (\rho u \Delta y)_w$$

$$\boxed{a_E > 0 \;\; if \;\; \frac{(\rho u \Delta y)_e}{2} > \Gamma\frac{\Delta y_w}{\delta x_w} \Leftrightarrow \frac{(\rho u \delta x_w)_e}{\Gamma} > 2}$$

UPWIND

$$a_E = -\Gamma\frac{S_e}{\delta x_e}$$

$$a_W = -\Gamma\frac{S_w}{\delta x_w} - (\rho u \Delta y)_w$$

$$a_P = \Gamma\frac{S_e}{\delta x_e} + \Gamma\frac{S_w}{\delta x_w} + (\rho u \Delta y)_e$$
$$= -(a_E + a_W) + (\rho u \Delta y)_e - (\rho u \Delta y)_w$$

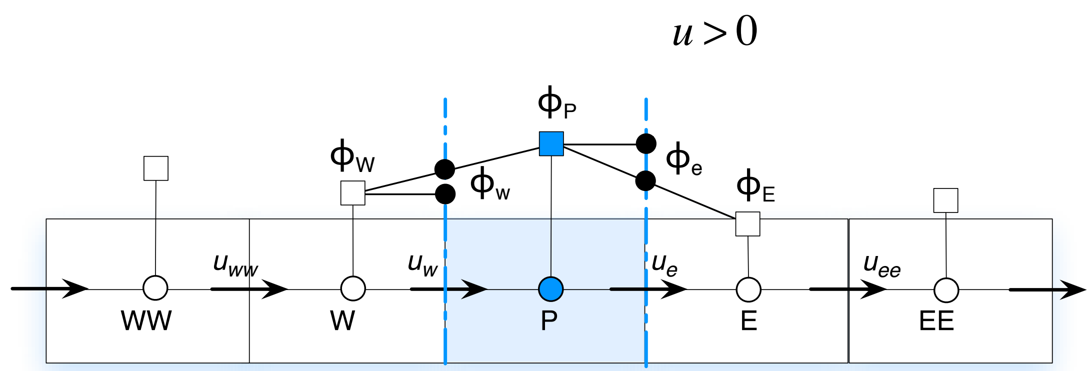
$$\boxed{a_E, a_W < 0}$$

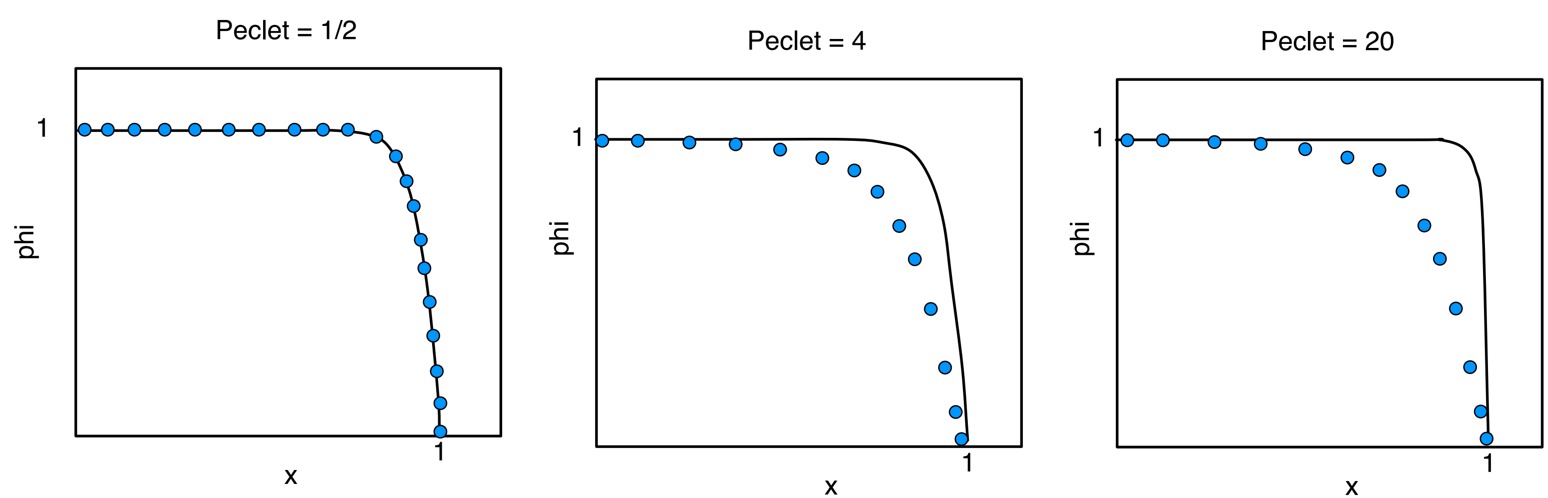
Hybrid Scheme

$$a_E = \begin{cases} -\Gamma \dfrac{\Delta y_e}{\delta x_e} + \dfrac{(\rho u \Delta y)_e}{2}, if \;\; Pe_e < 2 \\ -\Gamma \dfrac{\Delta y_e}{\delta x_e}, otherwise \end{cases}$$

$$a_W = \begin{cases} -\Gamma \dfrac{\Delta y_w}{\delta x_w} - \dfrac{(\rho u \Delta y)_w}{2}, if \;\; Pe_w < 2 \\ -\Gamma \dfrac{\Delta y_w}{\delta x_w} - (\rho u \Delta y)_w, otherwise \end{cases}$$

$$a_P = -(a_E + a_W) + (\rho u \Delta y)_e - (\rho u \Delta y)_w$$





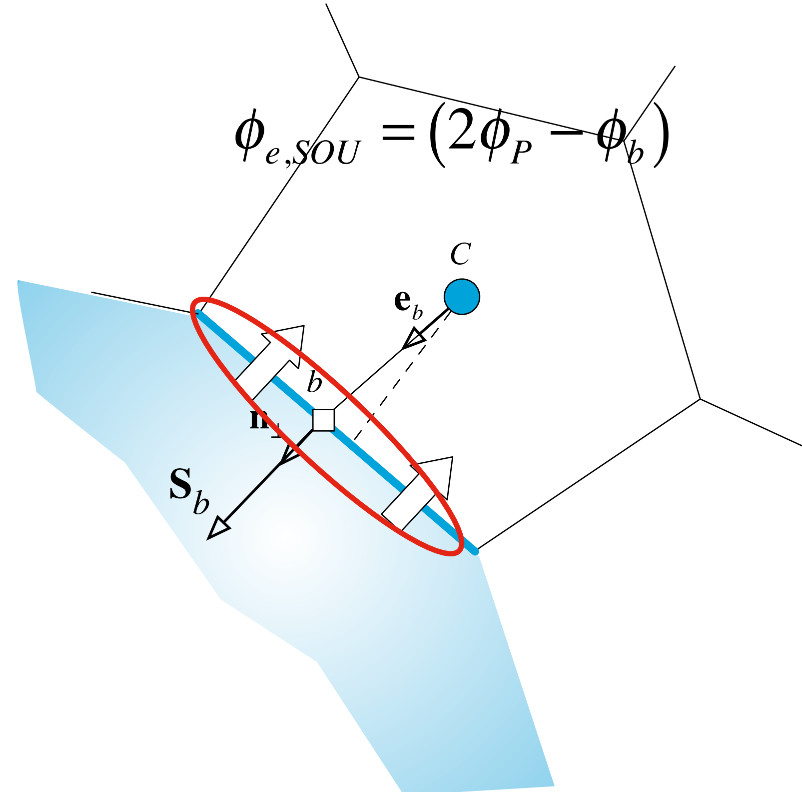
Boundary Conditions

Inlet BC

$$\sum_{f=nb(P)} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} = 0$$

$$\sum_{f=e,w,n} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} + (\rho_b \mathbf{v}_b \cdot \mathbf{S}_b)\phi_b = 0$$

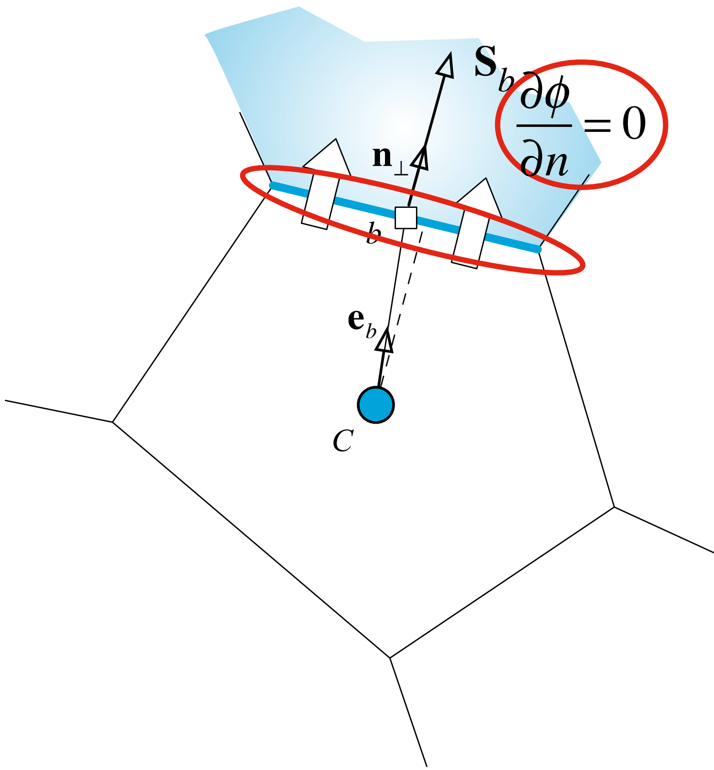
$$\sum_{f=e,w,n} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} = -(\rho_b \mathbf{v}_b \cdot \mathbf{S}_b)\phi_b$$



$$\phi_{e,SOU} = (2\phi_P - \phi_b)$$

$$a_P \phi_P + \sum_{N=E,N,S} a_N \phi_N = \underbrace{-(\rho_e \mathbf{v}_b \cdot \mathbf{S}_b)\phi_b}_{a_b \phi_b} - \underbrace{\sum_{f=e,w,n} (\rho_f \mathbf{v_f} \cdot \mathbf{S_f})(\phi_{f,SOU} - \phi_{f,UPWIND})}_{b_P^{dc}}$$

Outlet BC



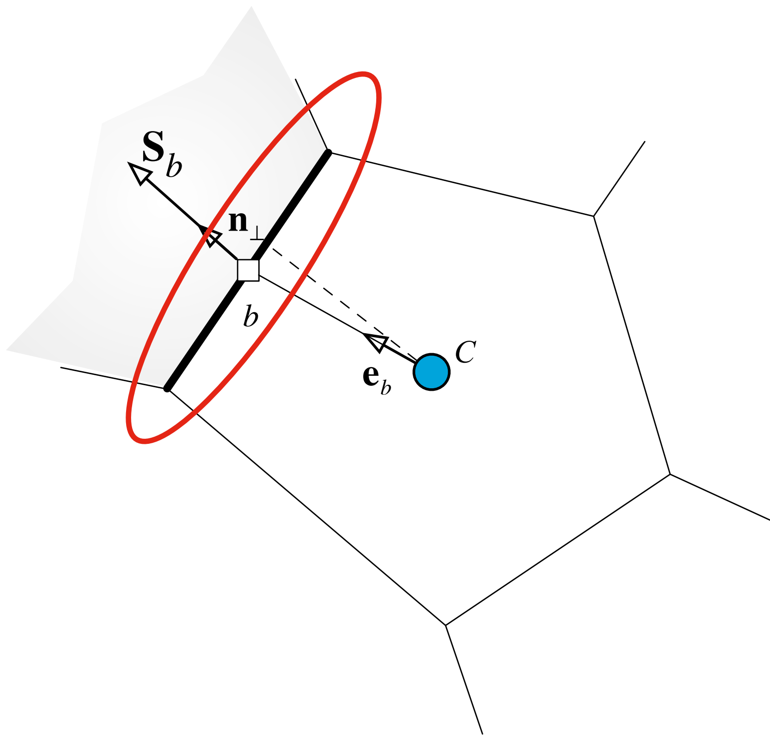
$$\sum_{f=nb(P)} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} = 0$$

$$\sum_{f=e,w,n} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} + (\rho_b \mathbf{v}_b \cdot \mathbf{S}_b)\phi_b = 0$$

$$\sum_{f=e,w,n} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} + (\rho_b \mathbf{v}_b \cdot \mathbf{S}_b)\phi_P = 0$$

$$a_P \phi_P + \sum_{N=W,N,S} a_N \phi_N = \underbrace{-\sum_{f=w,n,s} \left(\rho_f \mathbf{v_f} \cdot \mathbf{S_f}\right)\left(\phi_{f,SOU} - \phi_{f,UPWIND}\right)}_{b_P^{dc}}$$

Wall BC



$$\sum_{f=nb(P)} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} = 0$$

$$(\rho_b \mathbf{v}_b \cdot \mathbf{S}_b) = 0$$

$$\sum_{f=e,w,s} (\rho \mathbf{v} \phi)_f \cdot \mathbf{S_f} = 0$$

$$a_P \phi_P + \sum_{N=E,W,S} a_N \phi_N = \underbrace{-\sum_{f=w,n,s} \left(\rho_f \mathbf{v_f} \cdot \mathbf{S_f}\right)\left(\phi_{f,SOU} - \phi_{f,UPWIND}\right)}_{b_P^{dc}}$$

$$\phi_{s,SOU} = (2\phi_P - \phi_b)$$

$$\phi_{s,SOU} = \left(\frac{3}{2}\phi_S - \frac{1}{2}\phi_{SS}\right)$$