
# Computing the Gradient

Chapter 09

# Gradient in Cartesian Grids

# Computing the Gradient

$$\left(\frac{\partial \phi}{\partial x}\right)_e = \frac{\phi_E - \phi_P}{x_E - x_P}$$

$$= \frac{\phi_E - \phi_P}{\delta x_e}$$

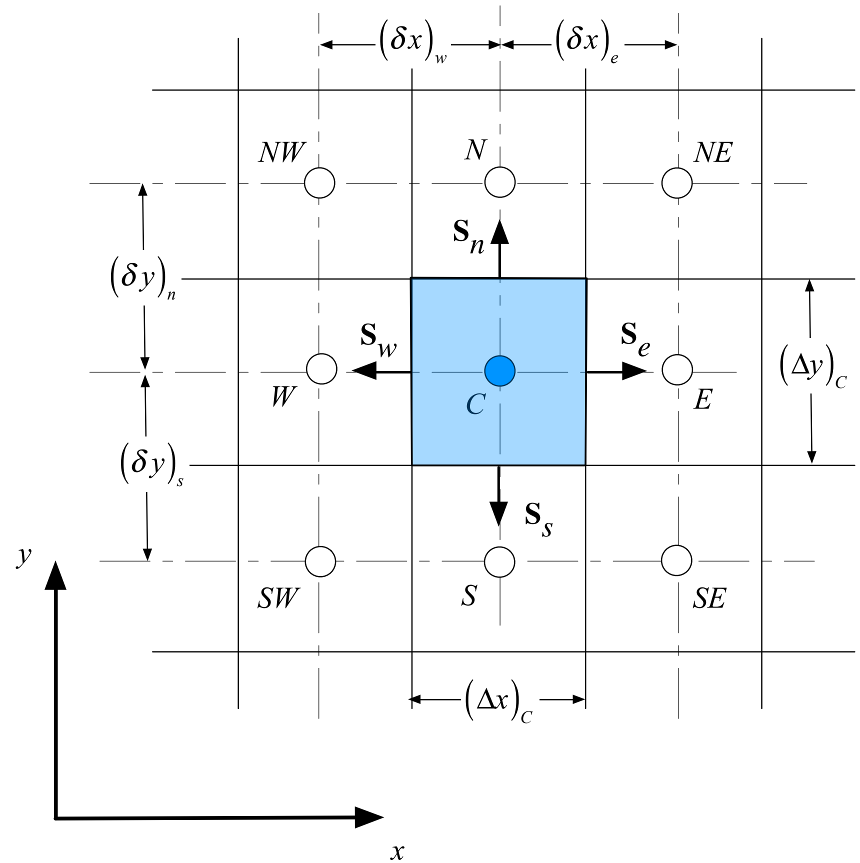
$$\left(\frac{\partial \phi}{\partial x}\right)_P = \frac{\phi_e - \phi_w}{x_e - x_w}$$

$$= \frac{\left(\frac{\phi_E + \phi_P}{2}\right) - \left(\frac{\phi_P + \phi_W}{2}\right)}{\Delta x_P}$$

$$= \frac{\phi_E - \phi_W}{2\Delta x_P}$$

$$\left(\frac{\partial \phi}{\partial x}\right)_P = \frac{\phi_E - \phi_W}{2\Delta x_P}$$

$$\left(\frac{\partial \phi}{\partial y}\right)_P = \frac{\phi_N - \phi_S}{2\Delta y_P}$$



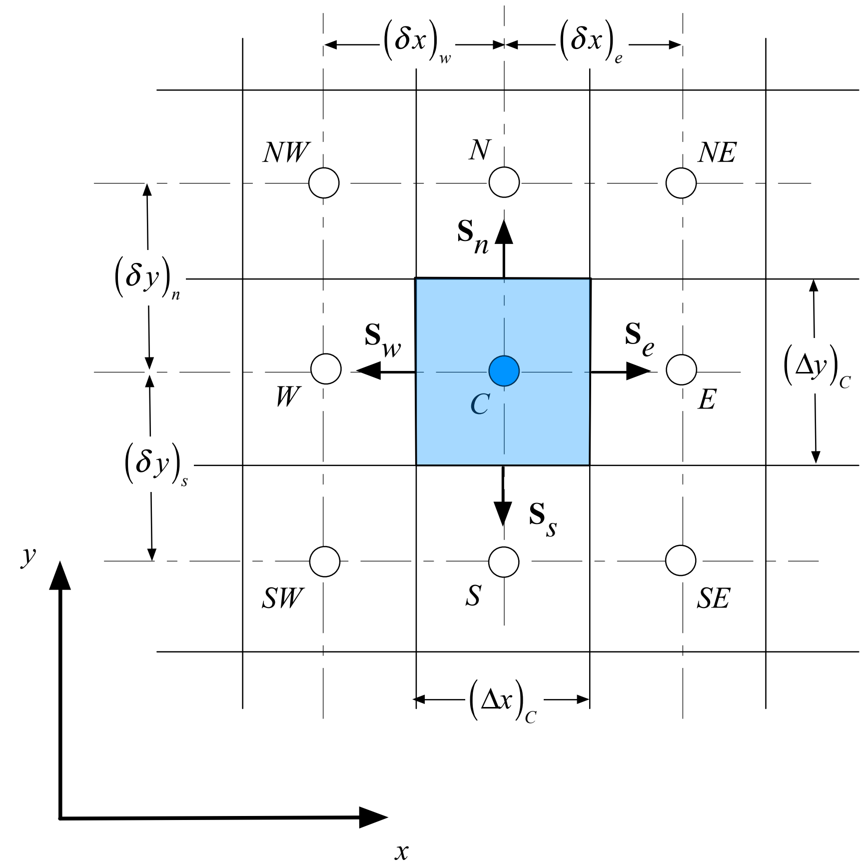
# Gauss Gradient

$$\int_{\Omega} \nabla \phi \, d\Omega = \oint_{\partial\Omega} \phi \, d\mathbf{S}$$

$$\overline{\nabla \phi} \, \Omega = \oint_{\partial\Omega} \phi \, d\mathbf{S}$$

$$\overline{\nabla \phi_P} = \frac{1}{\Omega_P} \int_{\Omega_P} \nabla \phi \, d\Omega = \frac{1}{\Omega_P} \oint_{\partial\Omega} \phi \, d\mathbf{S}$$

$$\overline{\nabla \phi_P} = \frac{1}{\Omega_P} \sum_{\sim f} \phi_f \mathbf{S}_f$$



# Computing the Gradient

$$\overline{\nabla \phi_P} = \frac{1}{\Omega_P} \sum_{\sim f} \phi_f \mathbf{S}_f$$

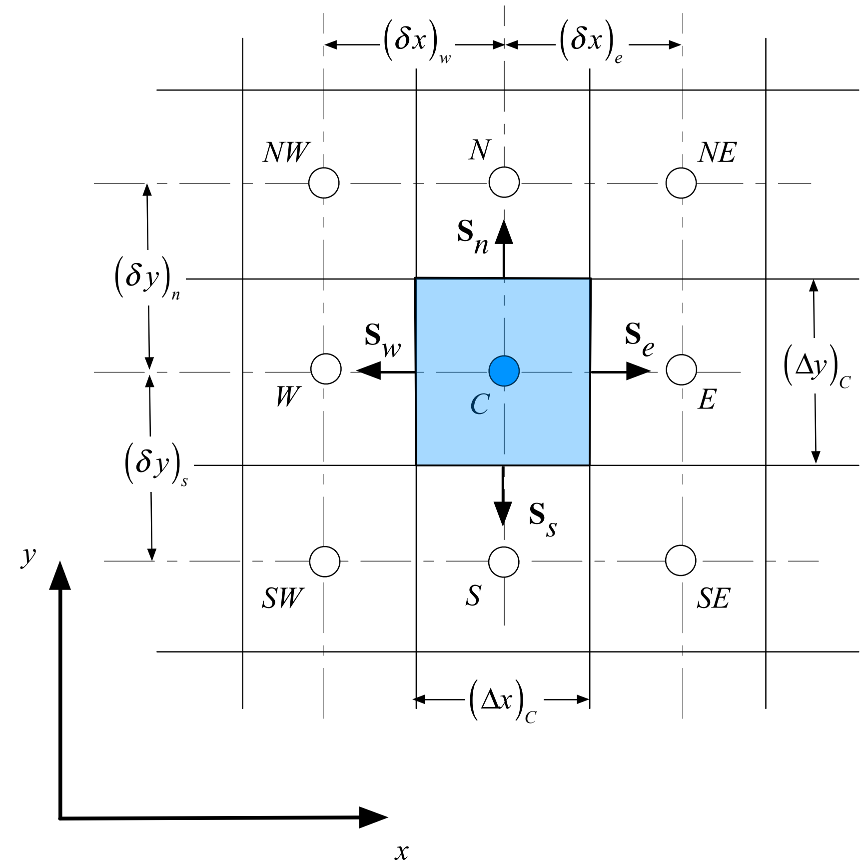
1. Compute face values over all faces

$$\phi_f = \phi_N \lambda_f + \phi_P \left(1 - \lambda_f\right)$$

2. Loop over faces and assemble the discrete integral

$$\phi_f \mathbf{S}_f$$

3. Divide the complete integrals at the center of the cells by the volume



# Gradient in Unstructured Grids

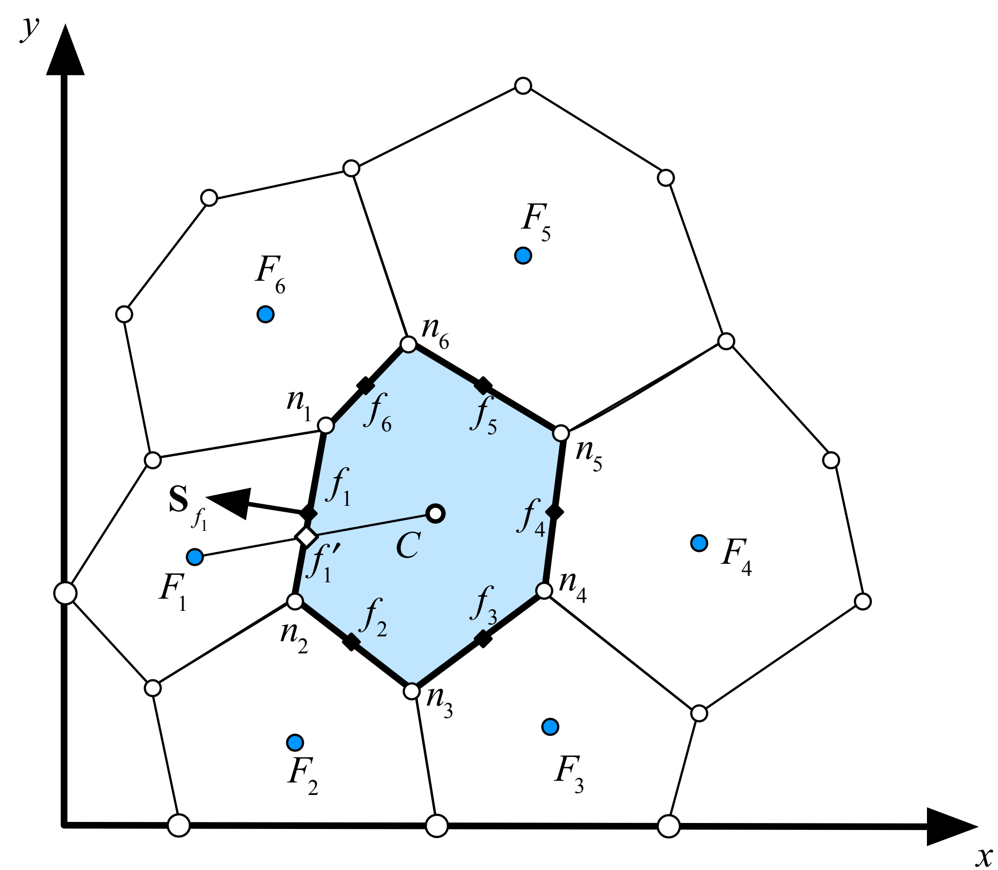
# Unstructured Grids

$$\int_{\Omega} \nabla \phi \, d\Omega = \oint_{\partial\Omega} \phi \, d\mathbf{S}$$

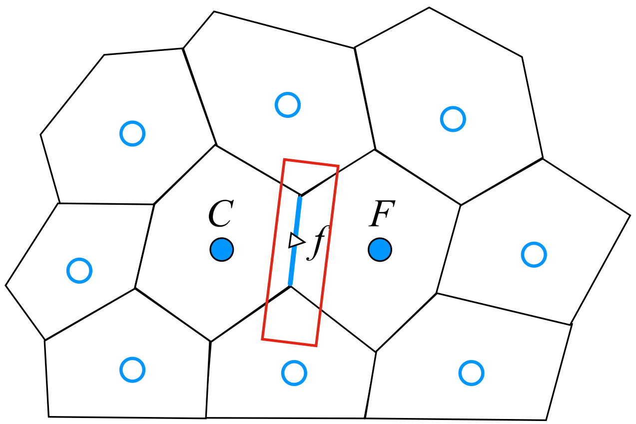
$$\overline{\nabla \phi}\, \Omega = \int_{\Omega} \nabla \phi \, d\Omega$$

$$\overline{\nabla \phi_P} = \frac{1}{\Omega_P} \int_{\Omega_P} \nabla \phi \, d\Omega = \frac{1}{\Omega_P} \oint_{\partial\Omega} \phi \, d\mathbf{S}$$

$$\overline{\nabla \phi_P} = \frac{1}{\Omega_P} \sum_{\sim f} \phi_f \mathbf{S}_f$$

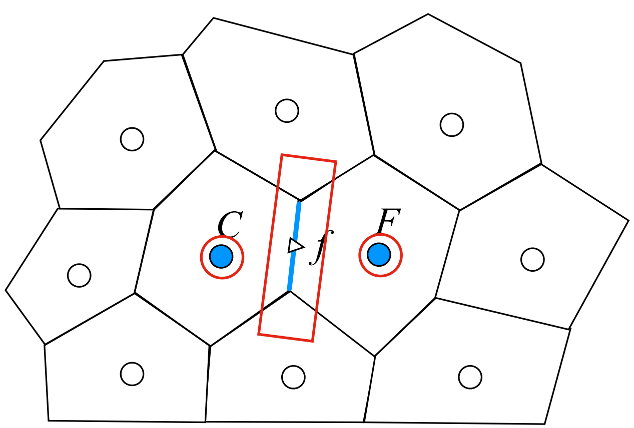


# Gradients at Cell Faces




$\overline{\nabla\phi_f}\cdot\mathbf{e}_{CF}$ stencil

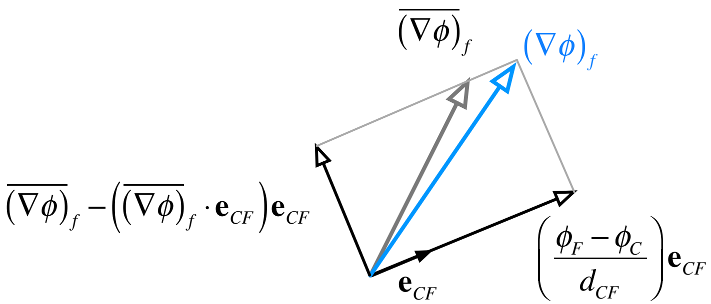
$$\overline{\nabla\phi_f} = \nabla\phi_N\lambda_f + \nabla\phi_P\left(1-\lambda_f\right)$$

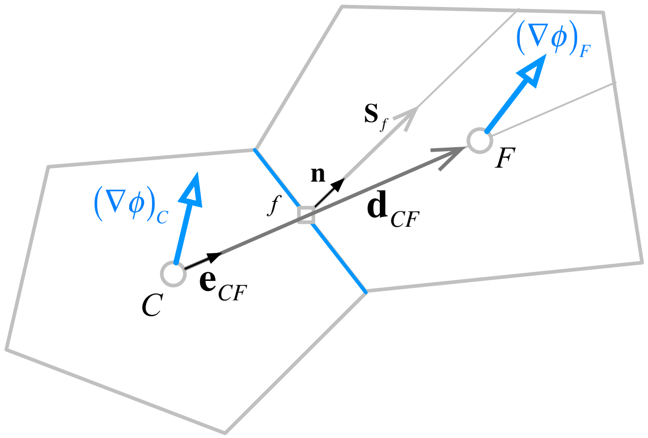


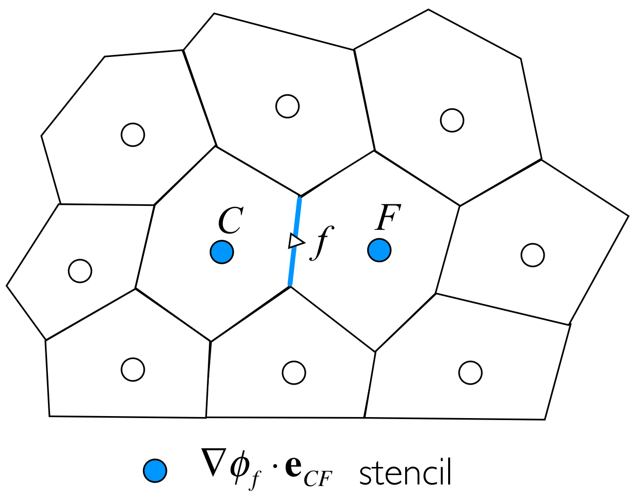

$\nabla\phi_f\cdot\mathbf{e}_{CF}$ stencil

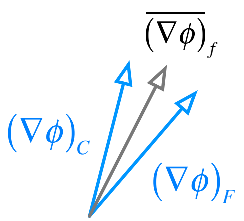
$$\nabla\phi_f = \overline{\nabla\phi_f} - \left(\overline{\nabla\phi_f}\cdot\mathbf{e}_{\mathbf{dc}}\right)\mathbf{e}_{\mathbf{dc}} + \frac{\phi_D-\phi_C}{\left\|d_{CD}\right\|}\mathbf{e}_{\mathbf{dc}}$$

# Interpolated Gradient



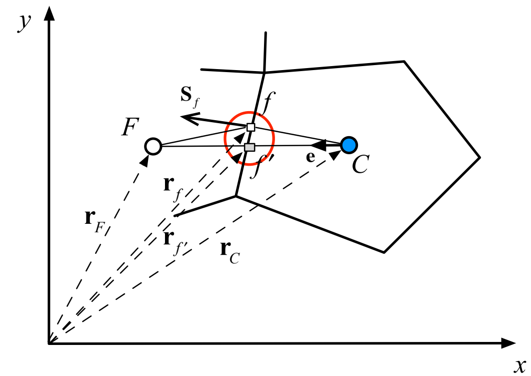



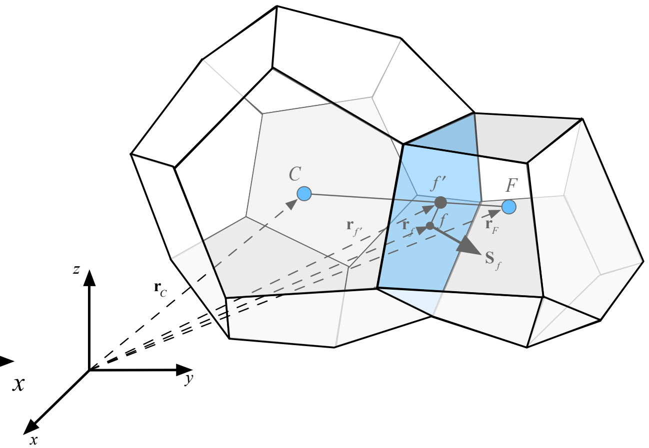
# Improved Gauss Gradient

# Non-Conjunctionality

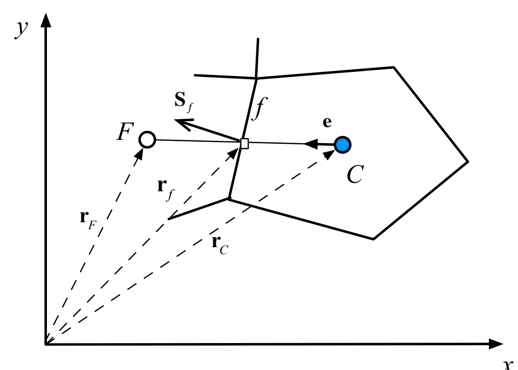
Non-Conjunctional Control Volume

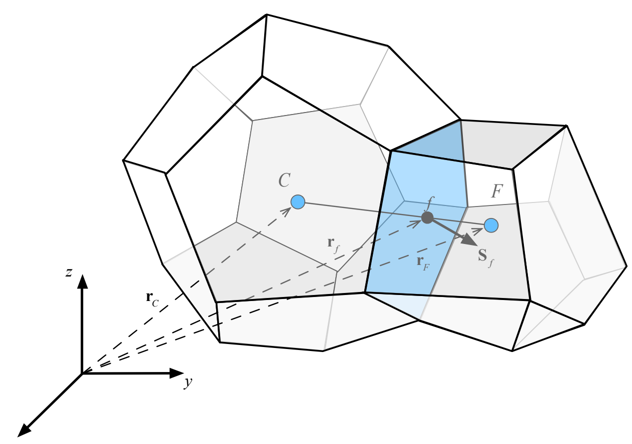
Difference between f and f' results in a decrease of the order of accuracy



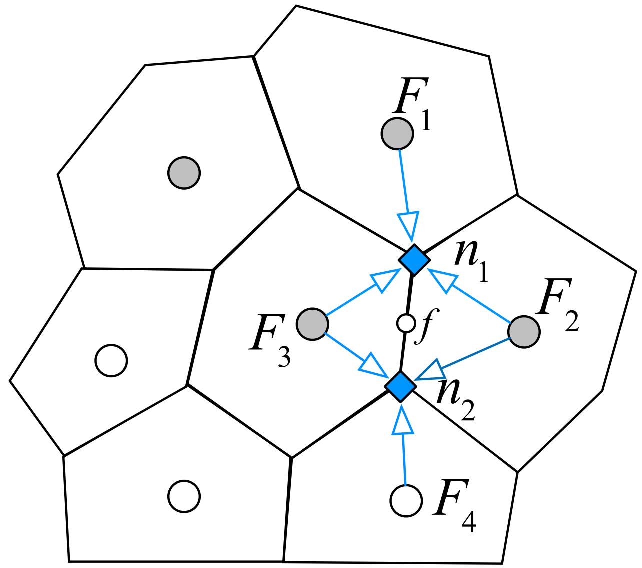




Conjunctional Control Volume



# Nodal Points

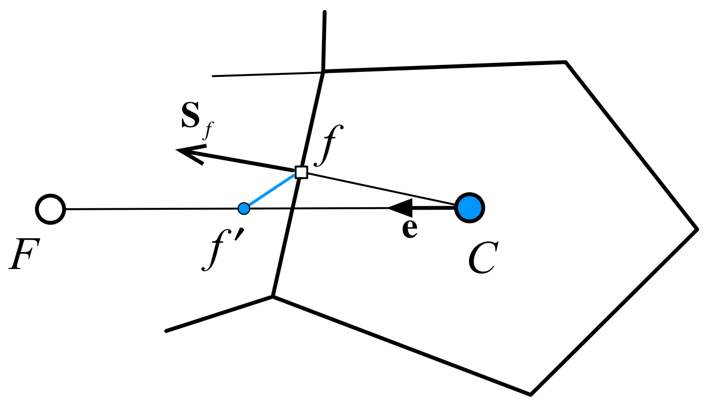


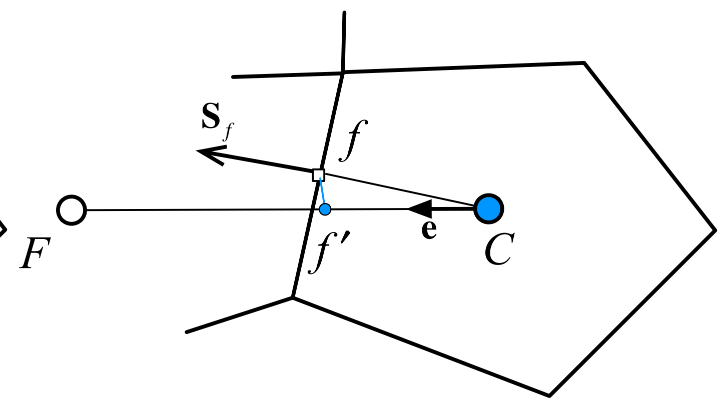

$$\phi_V = \frac{\displaystyle\sum_{n=1}^{nv(V)} \frac{\phi_{P_n}}{\left|\mathbf{r}_V - \mathbf{r}_{P_n}\right|}}{\displaystyle\sum_{n=1}^{nv(V)} \frac{1}{\left|\mathbf{r}_V - \mathbf{r}_{P_n}\right|}}$$

$$\nabla\phi_P = \frac{1}{\Omega_P}\sum_{f=1}^{nb}\phi_f \mathbf{S}_f$$

$$\nabla\phi_P = \frac{1}{\Omega_P}\sum_{f=1}^{nb}\frac{\phi_{f_{V1}} + \phi_{f_{V2}}}{2}\mathbf{S}_f$$

# Midpoint Rule




$$\begin{aligned}\phi_f &= \phi_{f'} + correction \\ &= \phi_{f'} + \underline{\overline{(\nabla\phi)}_{f'} \cdot \left(\mathbf{r}_f - \mathbf{r}_{f'}\right)}\end{aligned}$$

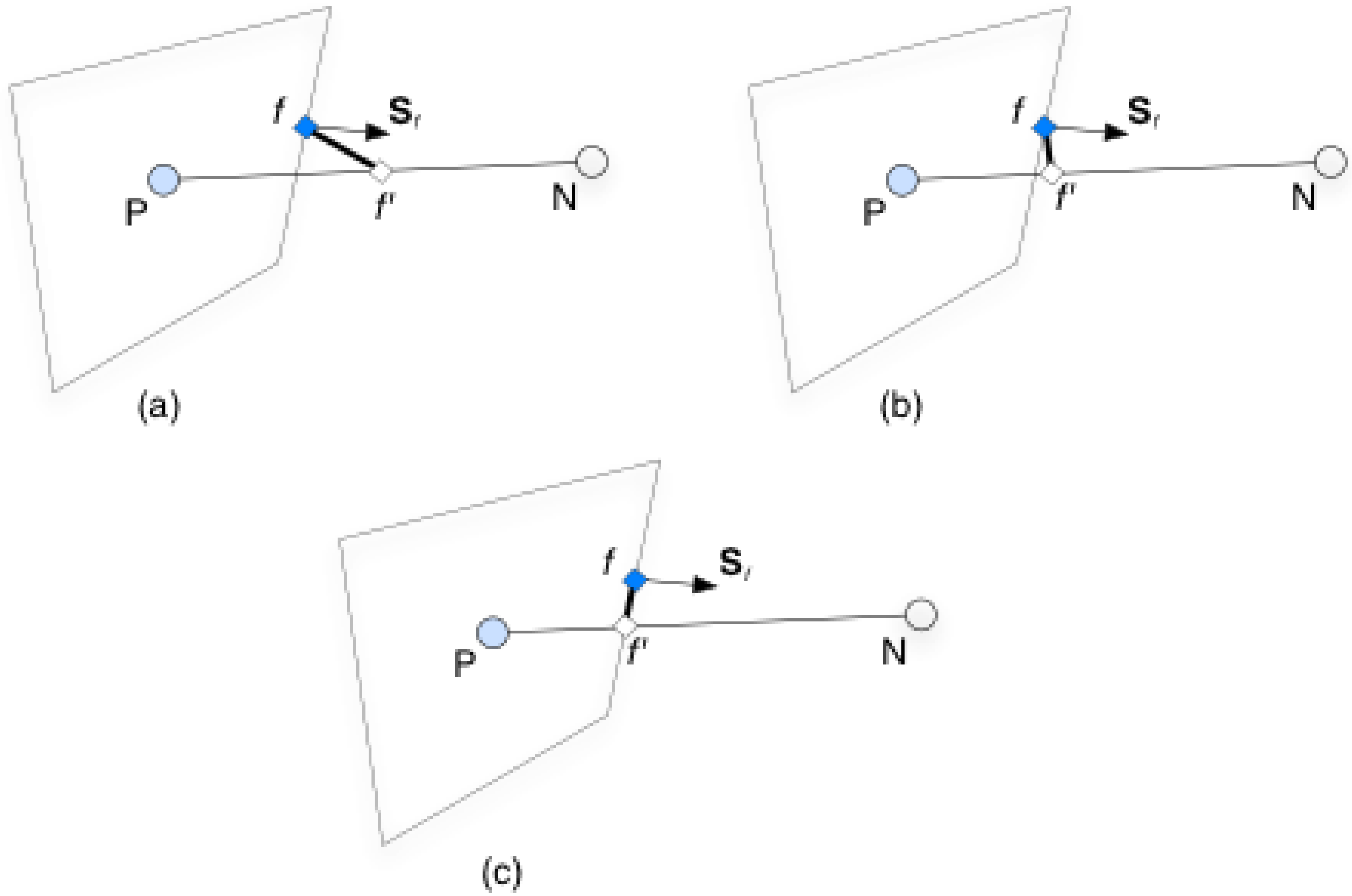
$$\breve{\nabla}\phi = \frac{1}{\Omega}\oint_f \phi_{f'}$$

$$\phi_f = \phi_{f'} + \left(\breve{\nabla}\phi\right)_{f'} \cdot \left(\mathbf{r}_f - \mathbf{r}_{f'}\right)$$

$$\nabla\phi \approx \frac{1}{\Omega}\oint_f \phi_f$$

$$\begin{aligned}\phi_f &= \phi_{f'} + \underline{correction} \\ &= \alpha\left\{\phi_P + (\nabla\phi)_P \cdot \left(\mathbf{r}_f - \mathbf{r}_P\right)\right\} + (1-\alpha)\left\{\phi_N + (\nabla\phi)_N \cdot \left(\mathbf{r}_f - \mathbf{r}_N\right)\right\} \\ &= \phi_{f'} + \underline{\alpha(\nabla\phi)_P \cdot \left(\mathbf{r}_f - \mathbf{r}_P\right) + (1-\alpha)(\nabla\phi)_N \cdot \left(\mathbf{r}_f - \mathbf{r}_N\right)}\end{aligned}$$

# Other Interpolations



# Least Square Gradient

# Least Square Error

$$\phi_F - \phi_P = (\nabla\phi)_P \cdot (\mathbf{r}_F - \mathbf{r}_P)$$

$$G_P = \sum_{k=1..NB(P)} \left\{ w_k \left[ (\nabla\phi)_P \cdot \left(\mathbf{r}_{F_k} - \mathbf{r}_P\right) - \left(\phi_{F_k} - \phi_P\right) \right]^2 \right\}$$

$$= \sum_{k=1..NB(P)} \left\{ w_k \left[ \Delta x_k \left(\frac{\partial\phi}{\partial x}\right)_P + \Delta y_k \left(\frac{\partial\phi}{\partial y}\right)_P - \Delta\phi_k \right]^2 \right\}$$

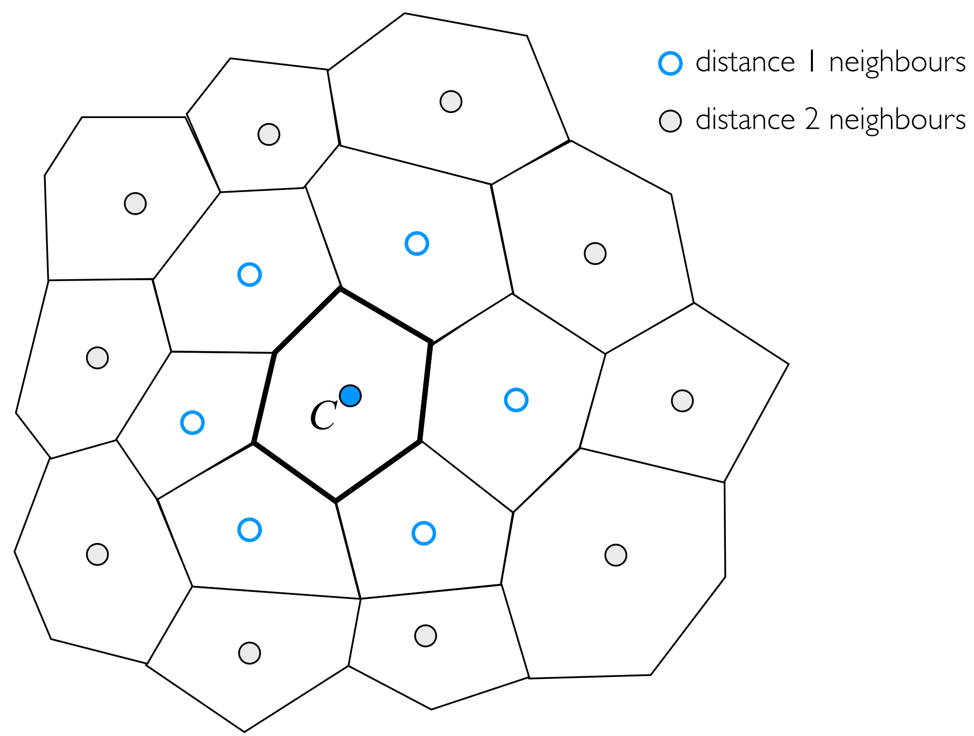
$$\Delta\phi_k = \phi_{F_k} - \phi_P$$

$$\Delta x_k = \left(\mathbf{r}_{F_k} - \mathbf{r}_P\right) \cdot \mathbf{i}$$

$$\Delta y_k = \left(\mathbf{r}_{F_k} - \mathbf{r}_P\right) \cdot \mathbf{j}$$

$$\frac{\partial G_P}{\partial \left(\dfrac{\partial\phi}{\partial x}\right)} = 0$$

$$\frac{\partial G_P}{\partial \left(\dfrac{\partial\phi}{\partial y}\right)} = 0$$



# Least Square Gradient

$$\sum_{k\sim NB(P)}\left\{2w_k\Delta x_k\left[\Delta x_k\left(\frac{\partial\phi}{\partial x}\right)_P+\Delta y_k\left(\frac{\partial\phi}{\partial y}\right)_P+\Delta z_k\left(\frac{\partial\phi}{\partial z}\right)_P-\Delta\phi_k\right]\right\}=0$$

$$\sum_{k\sim NB(P)}\left\{2w_k\Delta y_k\left[\Delta x_k\left(\frac{\partial\phi}{\partial x}\right)_P+\Delta y_k\left(\frac{\partial\phi}{\partial y}\right)_P+\Delta z_k\left(\frac{\partial\phi}{\partial z}\right)_P-\Delta\phi_k\right]\right\}=0$$

$$\sum_{k\sim NB(P)}\left\{2w_k\Delta z_k\left[\Delta x_k\left(\frac{\partial\phi}{\partial x}\right)_P+\Delta y_k\left(\frac{\partial\phi}{\partial y}\right)_P+\Delta z_k\left(\frac{\partial\phi}{\partial z}\right)_P-\Delta\phi_k\right]\right\}=0$$

$$\begin{bmatrix}\sum_{k\sim NB(P)}w_k\Delta x_k\Delta x_k & \sum_{k\sim NB(P)}w_k\Delta x_k\Delta y_k & \sum_{k\sim NB(P)}w_k\Delta x_k\Delta z_k\\ \sum_{k\sim NB(P)}w_k\Delta x_k\Delta y_k & \sum_{k\sim NB(P)}w_k\Delta y_k\Delta y_k & \sum_{k\sim NB(P)}w_k\Delta y_k\Delta z_k\\ \sum_{k\sim NB(P)}w_k\Delta z_k\Delta x_k & \sum_{k\sim NB(P)}w_k\Delta z_k\Delta y_k & \sum_{k\sim NB(P)}w_k\Delta z_k\Delta z_k\end{bmatrix}\begin{bmatrix}\left(\partial\phi/\partial x\right)\\ \left(\partial\phi/\partial y\right)\\ \left(\partial\phi/\partial z\right)\end{bmatrix}=\begin{bmatrix}\sum_{k\sim NB(P)}w_k\Delta x_k\Delta\phi_k\\ \sum_{k\sim NB(P)}w_k\Delta y_k\Delta\phi_k\\ \sum_{k\sim NB(P)}w_k\Delta z_k\Delta\phi_k\end{bmatrix}$$

# Least Square Gradient

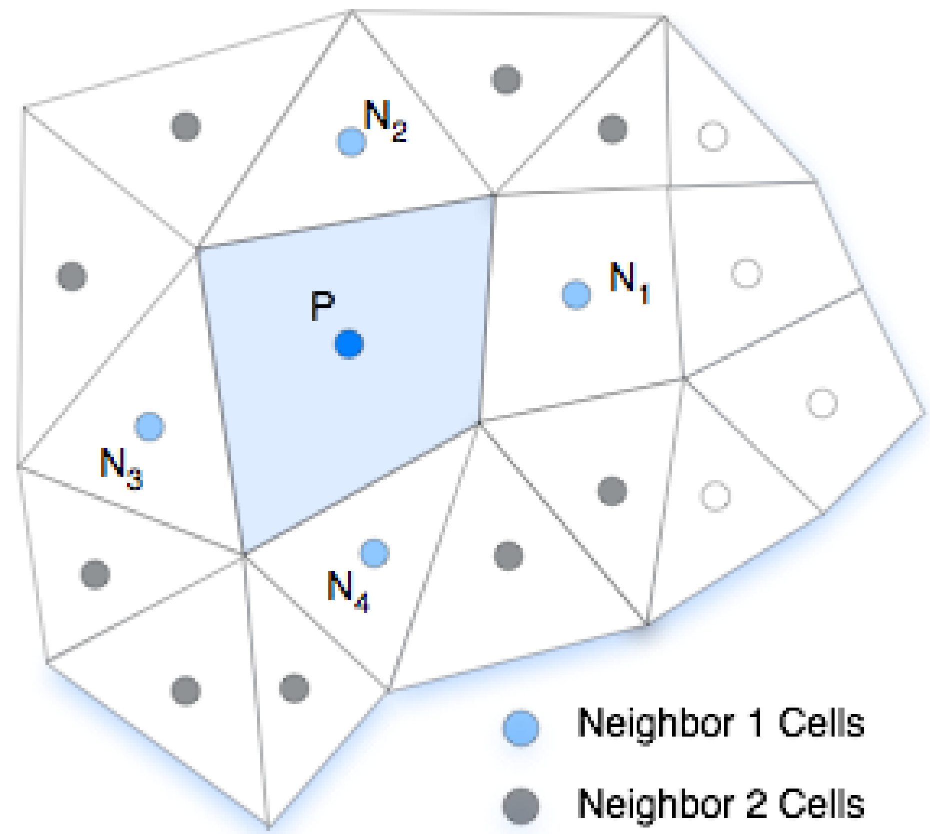
$$\left(\frac{\partial\phi}{\partial x}\right)_P = \frac{\sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta\phi_k \sum\limits_{k\sim NB(P)} w_k \Delta y_k \Delta y_k - \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta y_k \sum\limits_{k\sim NB(P)} w_k \Delta y_k \Delta\phi_k}{\sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta x_k \sum\limits_{k\sim NB(P)} w_k \Delta y_k \Delta y_k - \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta y_k \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta y_k}$$

*and*

$$\left(\frac{\partial\phi}{\partial x}\right)_y = \frac{\sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta x_k \sum\limits_{k\sim NB(P)} w_k \Delta y_k \Delta\phi_k - \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta y_k \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta\phi_k}{\sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta x_k \sum\limits_{k\sim NB(P)} w_k \Delta y_k \Delta y_k - \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta y_k \sum\limits_{k\sim NB(P)} w_k \Delta x_k \Delta y_k}$$

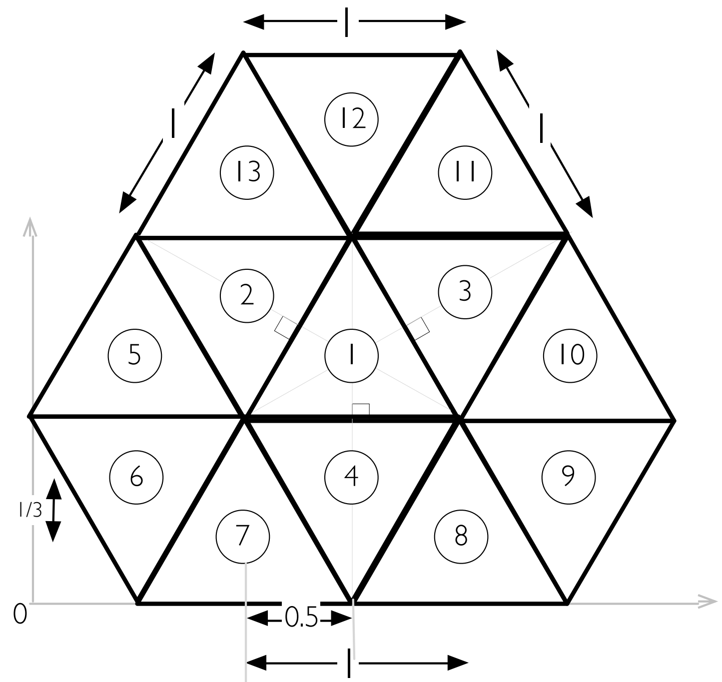
$$w_F = \frac{1}{\left\|\mathbf{r_F} - \mathbf{r_P}\right\|^2}$$

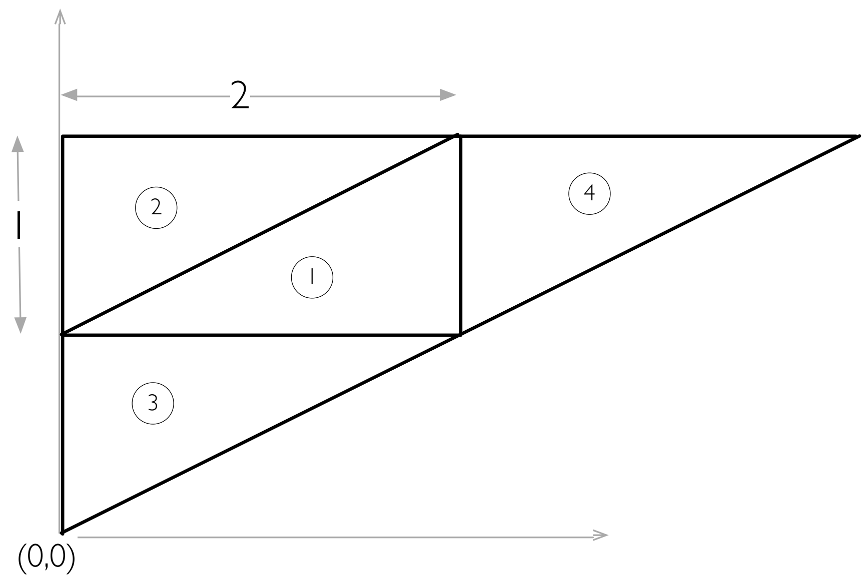
# Distance 2 Neighbours



# Gradient Interpolation

# Exercises


1
1
1
12
13
11
2
3
5
1
10
6
4
9
1/3
7
8
0
0.5
1


2
1
2
1
4
3
(0,0)