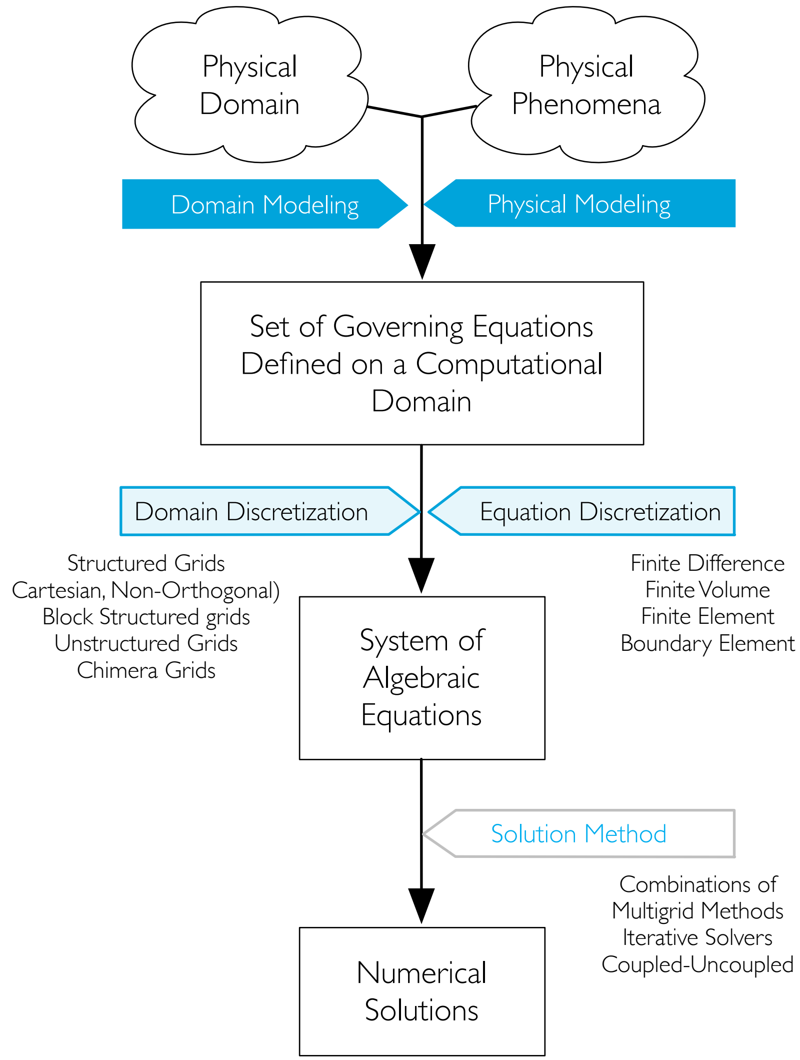
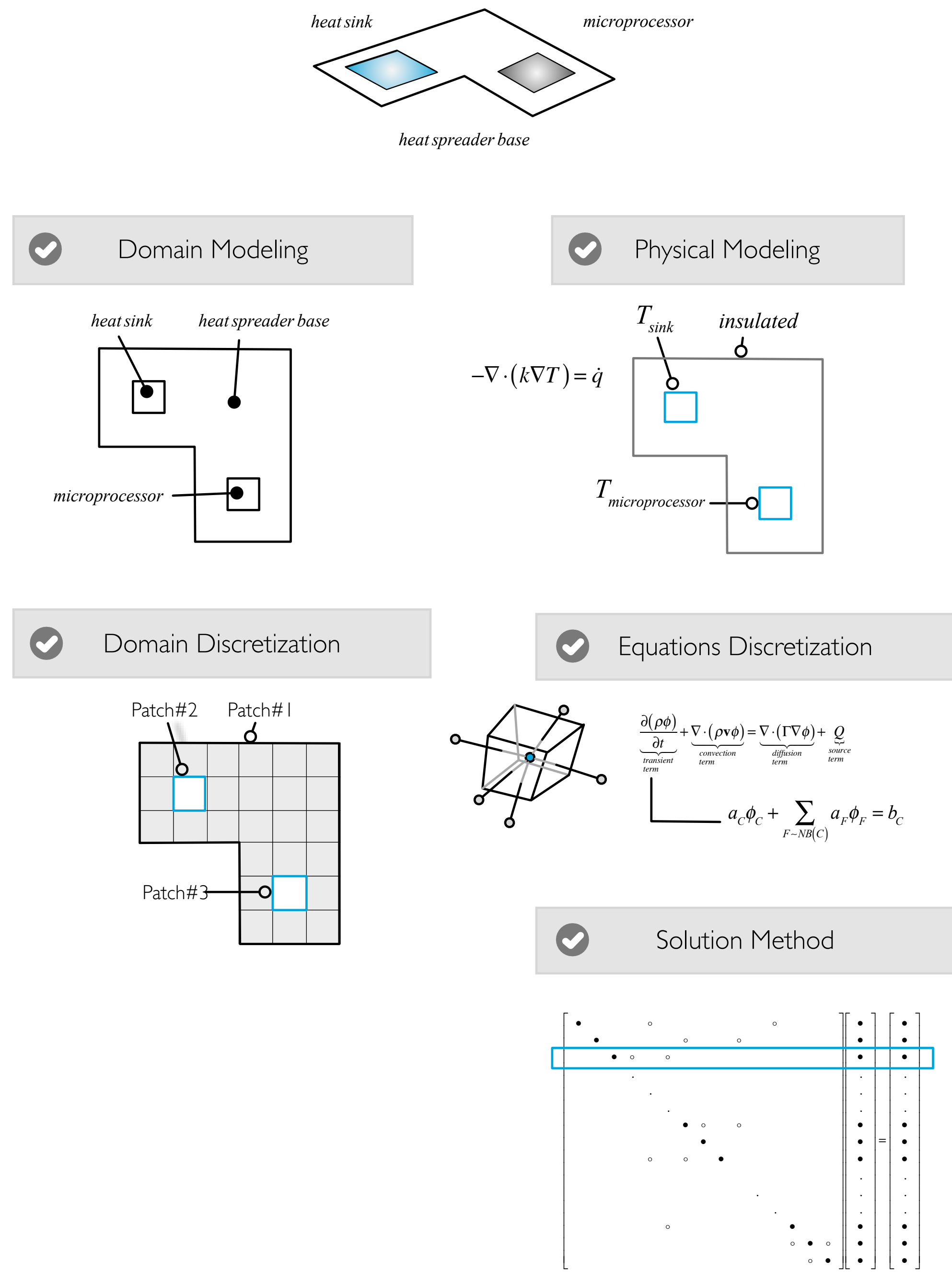
The Finite Volume Method

Chapter 05

The Process


Physical Domain
Physical Phenomena
Domain Modeling
Physical Modeling
Set of Governing Equations Defined on a Computational Domain
Domain Discretization
Equation Discretization
Structured Grids
Cartesian, Non-Orthogonal)
Block Structured grids
Unstructured Grids
Chimera Grids
Finite Difference
Finite Volume
Finite Element
Boundary Element
System of Algebraic Equations
Solution Method
Combinations of
Multigrid Methods
Iterative Solvers
Coupled-Uncoupled
Numerical Solutions


heat sink
microprocessor
heat spreader base
Domain Modeling
heat sink
heat spreader base
microprocessor
Physical Modeling
T_{sink}
insulated
$-\nabla \cdot (k \nabla T) = \dot{q}$
$T_{microprocessor}$
Domain Discretization
Patch#2
Patch#1
Patch#3
Equations Discretization
$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\text{transient term}} + \underbrace{\nabla \cdot (\rho \mathbf{v} \phi)}_{\text{convection term}} = \underbrace{\nabla \cdot (\Gamma \nabla \phi)}_{\text{diffusion term}} + \underbrace{Q}_{\text{source term}}$
$a_C \phi_C + \sum_{F \sim NB(C)} a_F \phi_F = b_C$
Solution Method

Balance Form

Consider the steady state conservation equation of a scalar

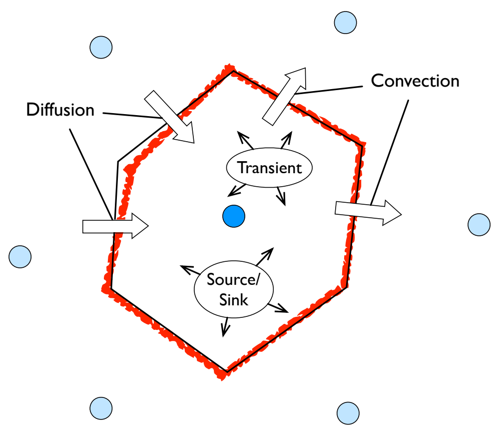
$$\underbrace{\nabla\cdot(\rho\mathbf{v}\phi)}_{convection\,term}=\underbrace{\nabla\cdot(\Gamma\nabla\phi)}_{diffusion\,term}+\underbrace{Q^{\phi}}_{source\,term}$$

$$\iint_V \nabla\cdot\mathbf{F}\,dV=\oint_{\partial V}\mathbf{F}\cdot d\mathbf{S}$$

$$\iint_V \nabla\cdot(\rho\mathbf{v}\phi)\,dV=\iint_V \nabla\cdot(\Gamma\nabla\phi)\,dV+\iint_V Q\,dV$$

$$\oint_{\partial V}(\rho\mathbf{v}\phi)\cdot d\mathbf{S}=\oint_{\partial V}(\Gamma\nabla\phi)\cdot d\mathbf{S}+\iint_V Q\,dV$$

$$\sum_{f=faces(V)}\int_f(\rho\mathbf{v}\phi)\cdot d\mathbf{S}=\sum_{f=faces(V)}\int_f(\Gamma\nabla\phi)\cdot d\mathbf{S}+\iint_V Q\,dV$$



Face Flux Integration

$$\Downarrow$$

$$\sum_{f=faces(V)} \int_f (\rho \mathbf{v} \phi) \cdot d\mathbf{S}$$

$$\int_f \mathbf{J} \cdot d\mathbf{S} = \sum_{ip(f)} (\mathbf{J} \cdot \mathbf{S})_{ip} = \sum_{ip(f)} \left(\mathbf{J}_{ip} \cdot w_{ip} \mathbf{S}_f \right)$$

$$(\rho \mathbf{u} \phi)_f \cdot \mathbf{S}_f$$

$$FluxT_f = FluxC_f \phi_C + FluxF_f \phi_{F_1} + FluxV_f$$

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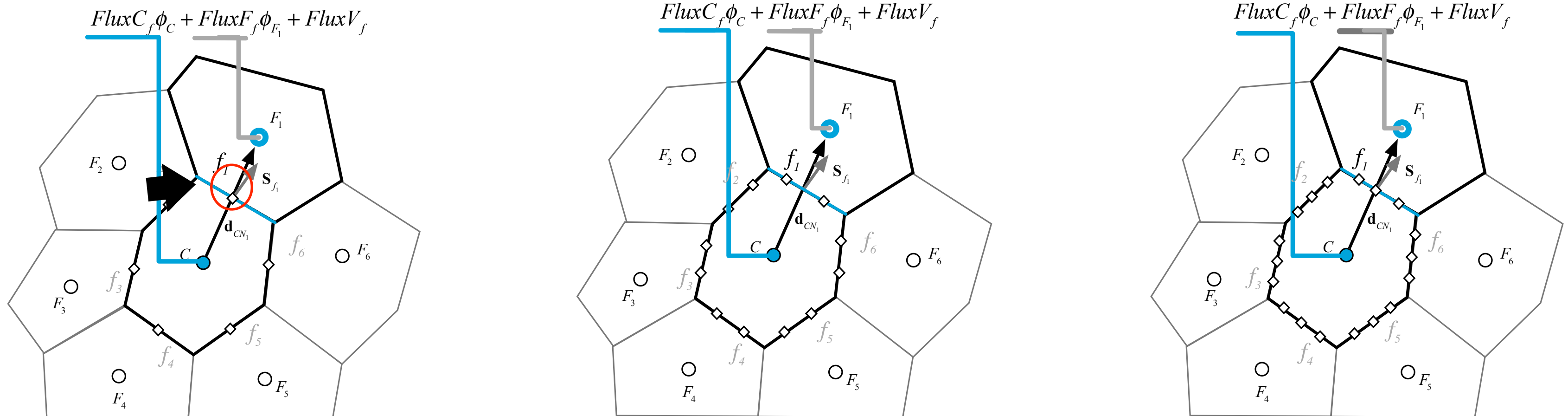


one integration point

two integration points

three integration points

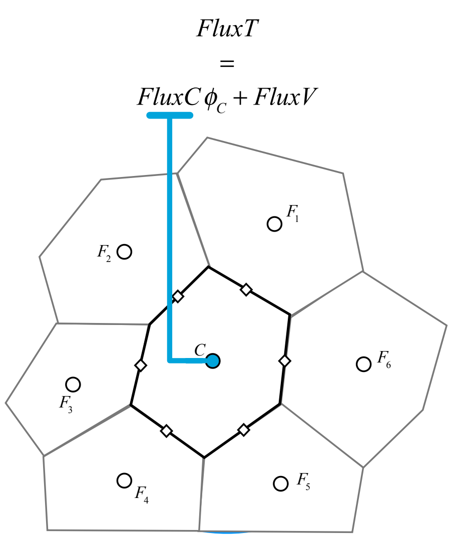
$$(\rho \mathbf{u} \phi)_{1(f)} \cdot w_{1(f)} \mathbf{S}_f + (\rho \mathbf{u} \phi)_{2(f)} \cdot w_{2(f)} \mathbf{S}_f$$

Source Integration

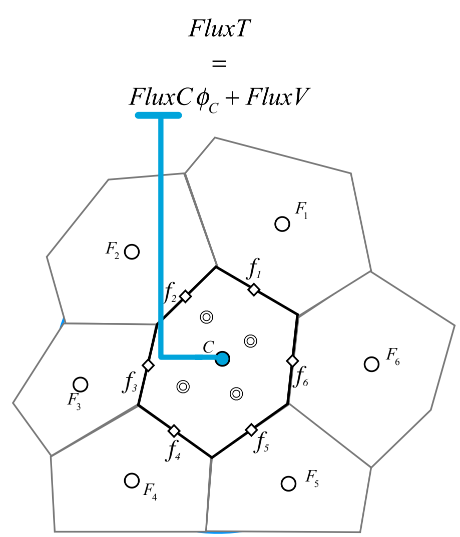
$$\color{green}{FluxT} = \iint_{V_C} Q\,dV = \sum_{ip(C)} Q_{C_{ip}} V_{C_{ip}} = \sum_{ip(C)} \left(Q_{C_{ip}} \omega_{ip} \right) V_C \qquad = \color{blue}{FluxC}\,\phi_C + FluxV$$

$$Q_C V_C$$

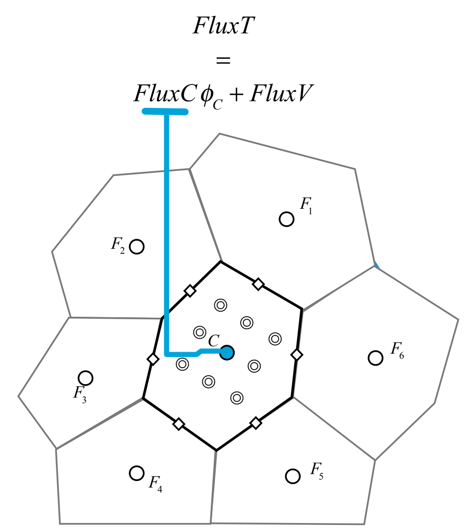
$$Q_{C_1} V_{C_1} + Q_{C_2} V_{C_2} + Q_{C_3} V_{C_3} + Q_{C_4} V_{C_4}$$



one integration point

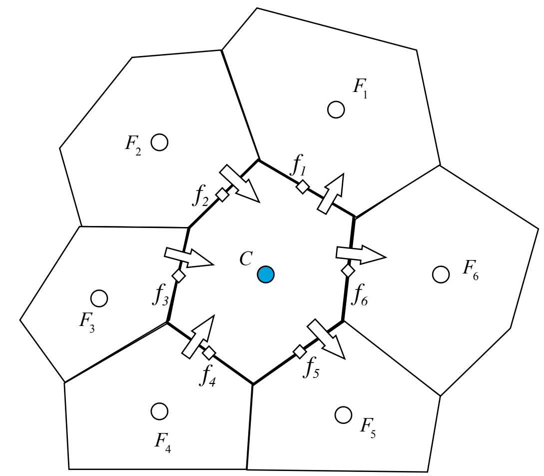


four integration points



nine integration points

Flux Linearization



$$\mathbf{J}_f^{\phi} \cdot \mathbf{S}_f = \underbrace{FluxT_f}_{\substack{total\ flux \\ for\ face\ f}} = \underbrace{FluxC_f}_{\substack{flux\ linearization \\ coefficient\ for\ C}} \phi_C + \underbrace{FluxF_f}_{\substack{flux\ linearization \\ coefficient\ for\ F}} \phi_F + \underbrace{FluxV_f}_{non-linearized\ part}$$

$$\sum_{f\sim nb(C)} \left(\mathbf{J}_f^{\phi} \cdot \mathbf{S}_f\right) = \sum_{f\sim nb(C)} \left(FluxT_f\right)$$

$$= \sum_{f\sim nb(C)} \left(FluxC_f\, \phi_C + FluxF_f\, \phi_F + FluxV_f\right)$$

$$Q_C^{\phi} V_C = FluxT$$

$$= FluxC\, \phi_C + FluxV$$

$$a_C = \sum_{f\sim nb(C)} FluxC_f - FluxC$$

$$a_C \phi_C + \sum_{F\sim NB(C)} \left(a_F \phi_F\right) = b_C$$

$$a_F = FluxF_f$$

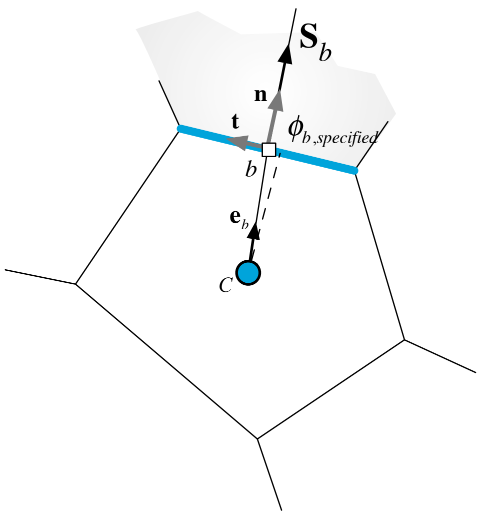
$$b_C = -\sum_{f\sim nb(C)} FluxV_f + FluxV$$

Boundary Conditions

Value Specified (Dirichlet Boundary Condition)

$$\begin{aligned}\mathbf{J}_b^{\phi}\cdot\mathbf{S}_b &= \mathbf{J}_b^{\phi,C}\cdot\mathbf{S}_b\\ &= \left(\rho\mathbf{v}\phi\right)_b\cdot\mathbf{S}_b\\ &= FluxC_b\phi_C + FluxV_b\\ &= \left(\rho_b\mathbf{v}_b\cdot\mathbf{S}_b\right)\phi_b = \dot{m}_f\phi_{b,specified}\end{aligned}$$

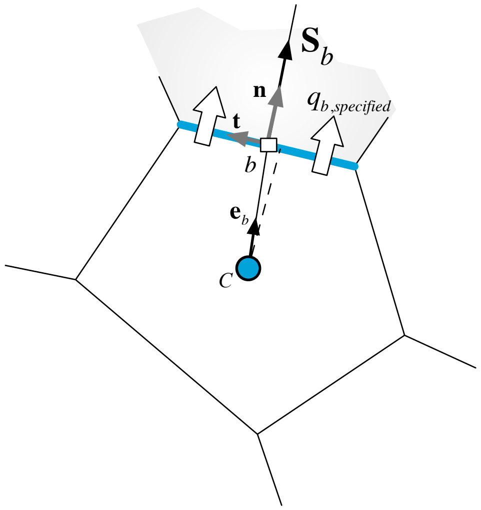
$$FluxC_b = 0$$
$$FluxV_b = \dot{m}_f\phi_{b,specified}$$



Flux Specified (Neumann Boundary Condition)

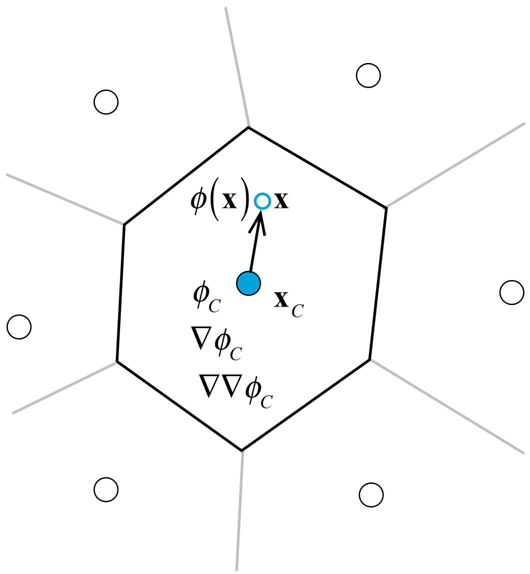
$$\begin{aligned}\mathbf{J}_b^{\phi}\cdot\mathbf{S}_b &= \underbrace{\mathbf{J}_b^{\phi}\cdot\mathbf{n}_b}_{Specified\ flux} S_b\\ &= q_{b,\,specified}S_b\end{aligned}$$

$$FluxC_b = 0$$
$$FluxV_b = q_{b,\,specified}S_b$$



FVM Accuracy

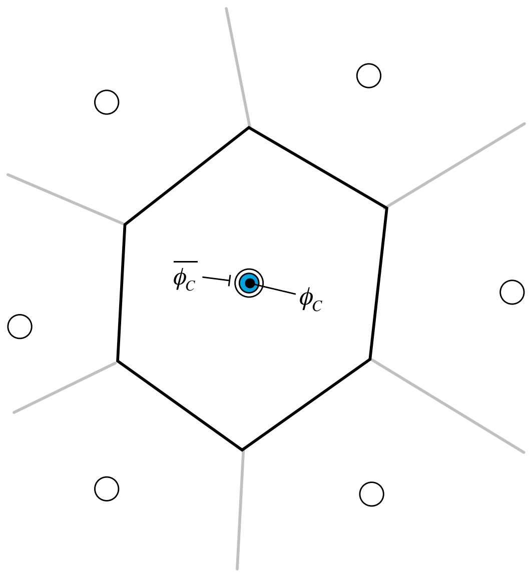
Spatial Variation



$$\phi(\mathbf{x}) = \phi_P + (\mathbf{x}-\mathbf{x}_P)\cdot(\nabla\phi)_P + \tfrac{1}{2}(\mathbf{x}-\mathbf{x}_P)^2 : (\nabla\nabla\phi)_P$$
$$+ \tfrac{1}{3!}(\mathbf{x}-\mathbf{x}_P)^3 :: (\nabla\nabla\nabla\phi)_P + \dots$$
$$+ \tfrac{1}{n!}(\mathbf{x}-\mathbf{x}_P)^n \underbrace{:::}_{n}(\underbrace{\nabla\nabla\nabla}_{n}\phi)_P + \dots$$

$$\phi(\mathbf{x}) = \phi_P + (\mathbf{x}-\mathbf{x}_P)\cdot(\nabla\phi)_P + O(\Delta x^2)$$

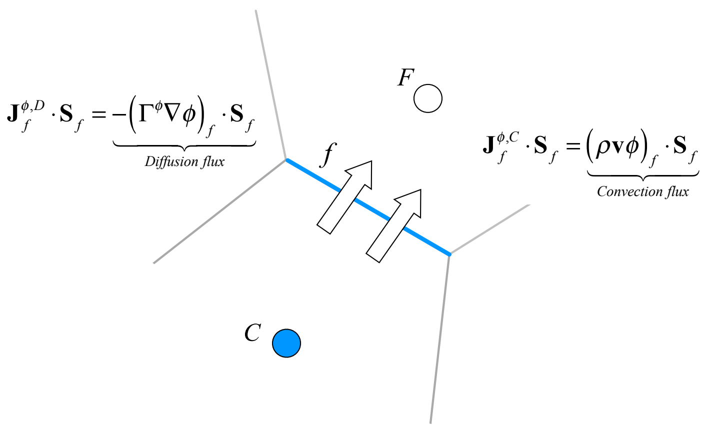
Mean Value Theorem



Cell Value flux

$$
\begin{aligned}
\overline{\phi}_C &= \frac{1}{V_C}\int_{V_C} \phi \, dV \\
&= \frac{1}{V_C}\int_{V_C} \left[\phi_C + (\mathbf{x}-\mathbf{x}_C)\cdot(\nabla\phi)_C + O\left(|\mathbf{x}-\mathbf{x}_C|^2\right) \right] dV \\
&= \frac{\phi_C}{V_C}\int_{V_C} dV + \frac{1}{V_C}\int_{V_C} (\mathbf{x}-\mathbf{x}_C)\cdot(\nabla\phi)_C \, dV + \frac{1}{V_C}\int_{V_C} O\left(|\mathbf{x}-\mathbf{x}_C|^2\right) dV \\
&= \phi_C + O\left(|\mathbf{x}-\mathbf{x}_C|^2\right)
\end{aligned}
$$

Mean Value Theorem



convective flux

$$\overline{\left(\rho_f \mathbf{v}_f \cdot \mathbf{S}_f\right)\phi_f} = \int_f (\rho\mathbf{v}\phi)\cdot d\mathbf{S}$$

$$= \int_f \rho_f \mathbf{v}_f \left[\phi_f + \left(\mathbf{x}-\mathbf{x}_f\right)\cdot(\nabla\phi)_f + O\left(\left|\mathbf{x}-\mathbf{x}_f\right|^2\right)\right]\cdot d\mathbf{S}$$

$$= \rho_f\mathbf{v}_f\phi_f\cdot\int_f d\mathbf{S} + \int_f \left(\mathbf{x}-\mathbf{x}_f\right)\cdot(\nabla\phi)_f\, \rho_f\mathbf{v}_f\cdot d\mathbf{S} + \int_f O\left(\left|\mathbf{x}-\mathbf{x}_f\right|^2\right)\rho_f\mathbf{v}_f\cdot d\mathbf{S}$$

$$= \left(\rho_f\mathbf{v}_f\cdot\mathbf{S}_f\right)\left[\phi_f + O\left(\left|\mathbf{x}-\mathbf{x}_f\right|^2\right)\right]$$

diffusion flux

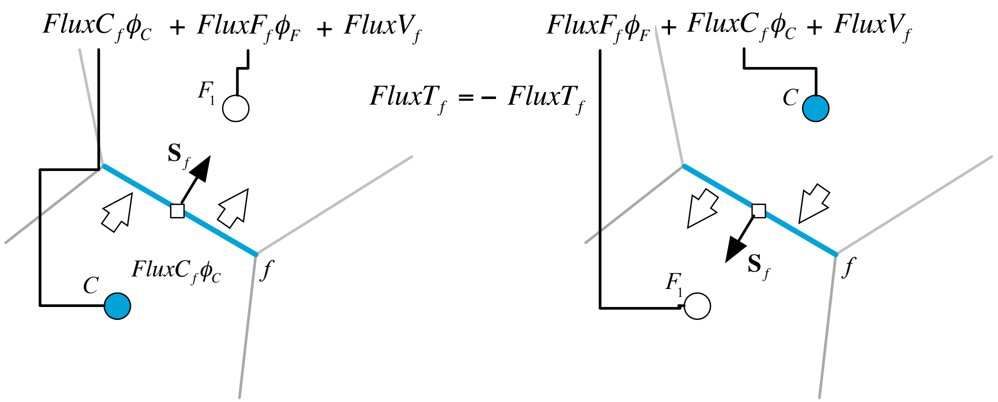
$$\int_f \left(\Gamma^{\phi}\nabla\phi\right)\cdot d\mathbf{S} = \int_f \left[\left(\Gamma^{\phi}\nabla\phi\right)_f + \left(\mathbf{x}-\mathbf{x}_f\right)\cdot\left(\nabla\left(\Gamma^{\phi}\nabla\phi\right)\right)_f + O\left(\left|\mathbf{x}-\mathbf{x}_f\right|^2\right)\right]\cdot d\mathbf{S}$$

$$= \left(\Gamma^{\phi}\nabla\phi\right)_f\cdot\int_f d\mathbf{S} + \left[\int_f \left(\mathbf{x}-\mathbf{x}_f\right)d\mathbf{S}\right]:\left(\nabla\left(\Gamma^{\phi}\nabla\phi\right)\right)_f + O\left(\left|\mathbf{x}-\mathbf{x}_f\right|^2\right)$$

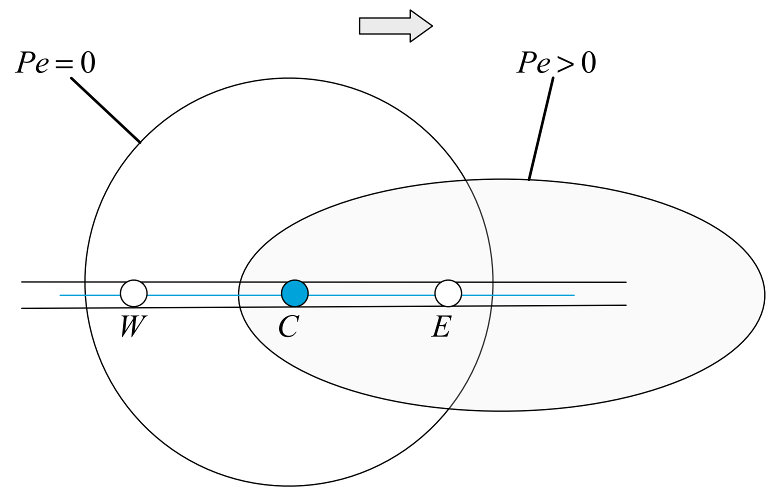
$$= \left(\Gamma^{\phi}\nabla\phi\right)_f\cdot\mathbf{S}_f + O\left(\left|\mathbf{x}-\mathbf{x}_f\right|^2\right)$$

Properties of the Discretized Equations

Conservation



Transportiveness



Other Properties

Consistency

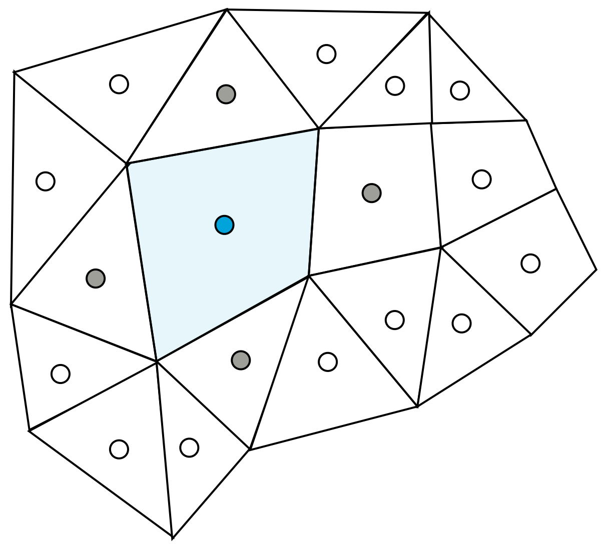
Stability

Economy

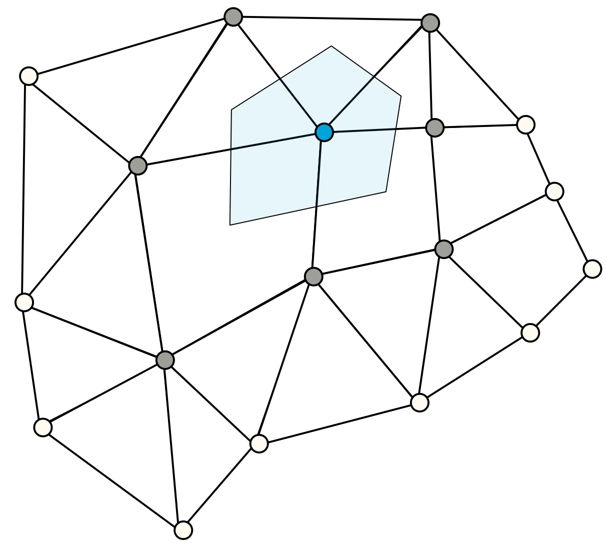
Boundedness of the Interpolation Profile

Variable Arrangment

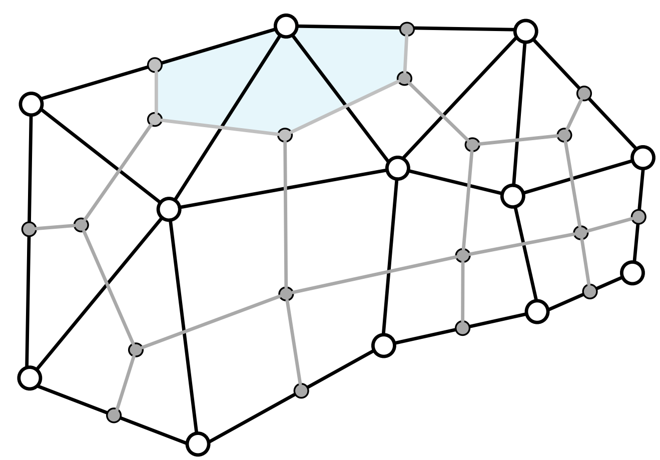
Variable Arrangement



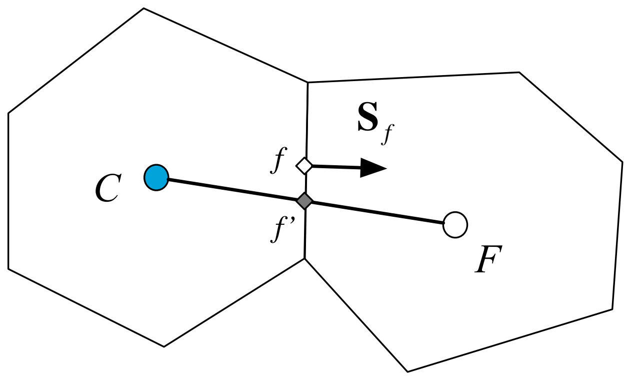
Cell-centered

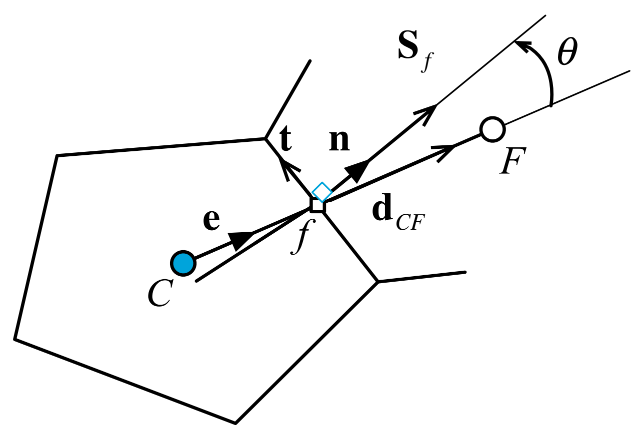


Vertex-centered



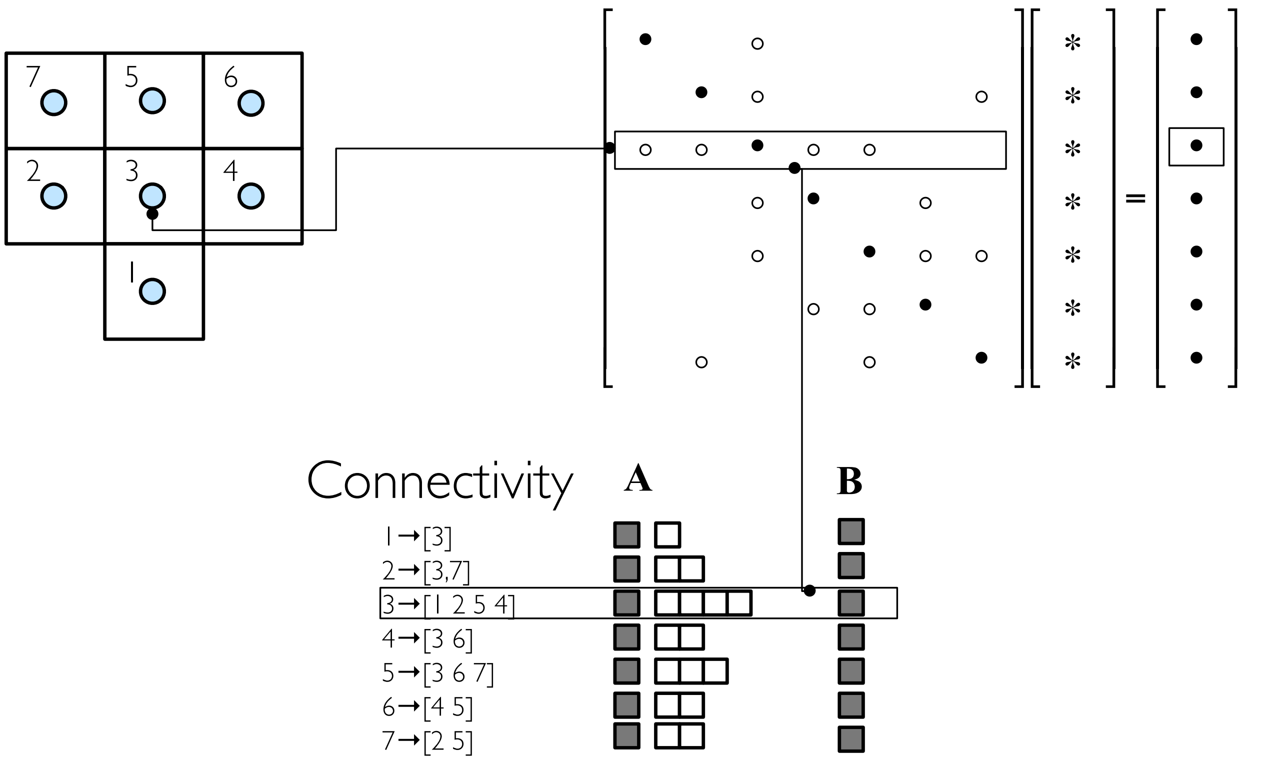
Other Issues





Matrix Connectivity

ufvm Connectivity



OpenFoam Connectivity