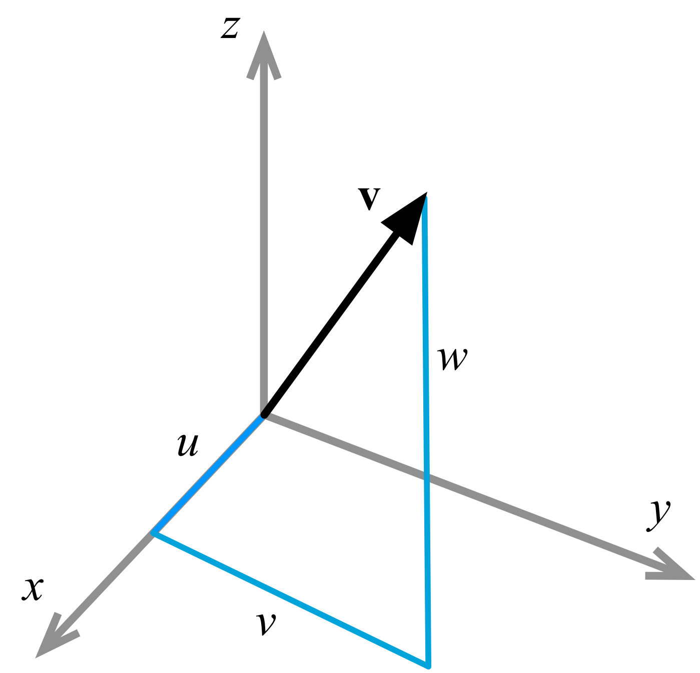
Mathematical Review

Chapter 02

Vector Operations

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad \mathbf{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \mathbf{v}^{\mathrm{T}} = \begin{bmatrix} u & v & w \end{bmatrix}$$

$$|\mathbf{v}| = \sqrt{u^2 + v^2 + w^2}$$

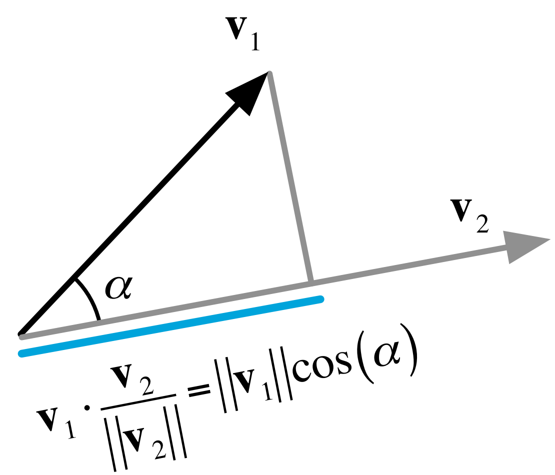


$$\mathbf{v}_1 = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} u_1 + u_2 \\ v_1 + v_2 \\ w_1 + w_2 \end{bmatrix}$$

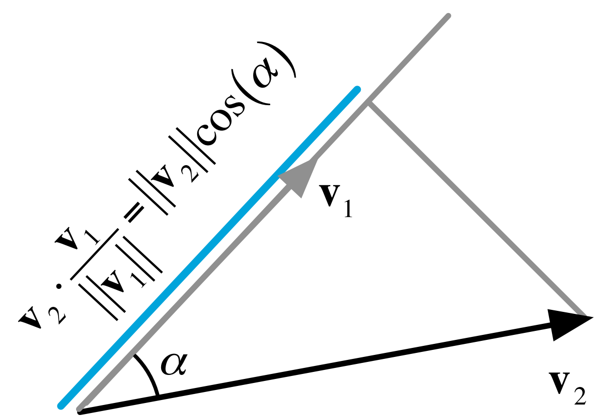
$$\begin{aligned} s\mathbf{v} &= s\left(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}\right) \\ &= su\mathbf{i} + sv\mathbf{j} + sw\mathbf{k} = \begin{bmatrix} su \\ sv \\ sw \end{bmatrix} \end{aligned}$$

Dot Product

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \cos(\mathbf{v}_1, \mathbf{v}_2)$$



$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$
$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = 0$$



$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= (u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}) \cdot (u_2\mathbf{i} + v_2\mathbf{j} + w_2\mathbf{k}) \\ &= u_1u_2 + v_1v_2 + w_1w_2 \end{aligned}$$

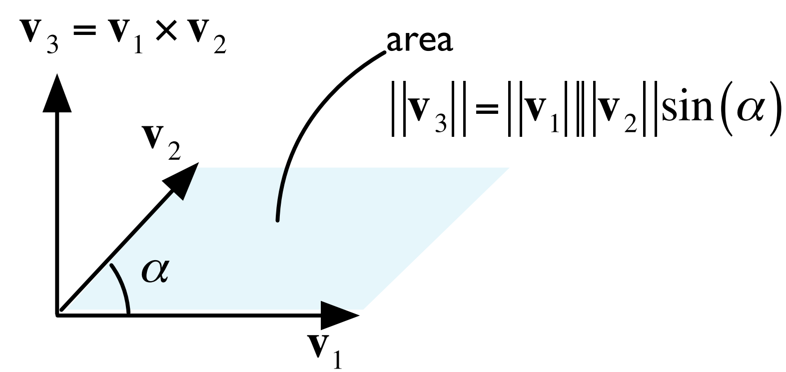
vector magnitude

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

unit vector

$$\mathbf{e}_v = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Cross Product



$$\|\mathbf{v}_3\| = \|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|\,\left|\sin(\mathbf{v}_1, \mathbf{v}_2)\right|$$

$$\begin{aligned}
\mathbf{v}_1 \times \mathbf{v}_2 &= (u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}) \times (u_2\mathbf{i} + v_2\mathbf{j} + w_2\mathbf{k}) \\
&= u_1u_2\mathbf{i}\times\mathbf{i} + u_1v_2\mathbf{i}\times\mathbf{j} + u_1w_2\mathbf{i}\times\mathbf{k} \\
&\quad + v_1u_2\mathbf{j}\times\mathbf{i} + v_1v_2\mathbf{j}\times\mathbf{j} + v_1w_2\mathbf{j}\times\mathbf{k} \\
&\quad + w_1u_2\mathbf{k}\times\mathbf{i} + w_1v_2\mathbf{k}\times\mathbf{j} + w_1w_2\mathbf{k}\times\mathbf{k} \\
&= u_1u_2 0 + u_1v_2\mathbf{k} + u_1w_2(-\mathbf{j}) \\
&\quad + v_1u_2(-\mathbf{k}) + v_1v_2 0 + v_1w_2\mathbf{i} \\
&\quad + w_1u_2\mathbf{j} + w_1v_2(-\mathbf{i}) + w_1w_2 0
\end{aligned}$$

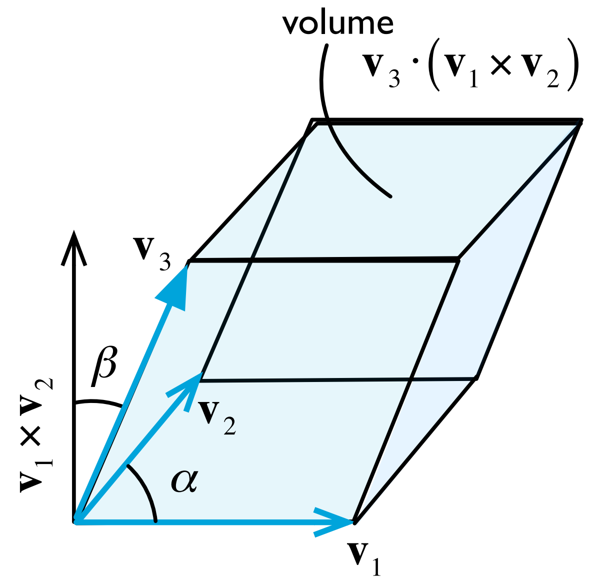
$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$
$$\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i}$$
$$\mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j}$$
$$\mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k}$$

determinant notation

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = \begin{bmatrix} v_1w_2 - v_2w_1 \\ u_2w_1 - u_1w_2 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Scalar Triple Product

$$\left(\mathbf{v}_1 \cdot \left[\mathbf{v}_2 \times \mathbf{v}_3\right]\right) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$



The scalar triple product represent the volume defined by the paralleled formed by the base vector

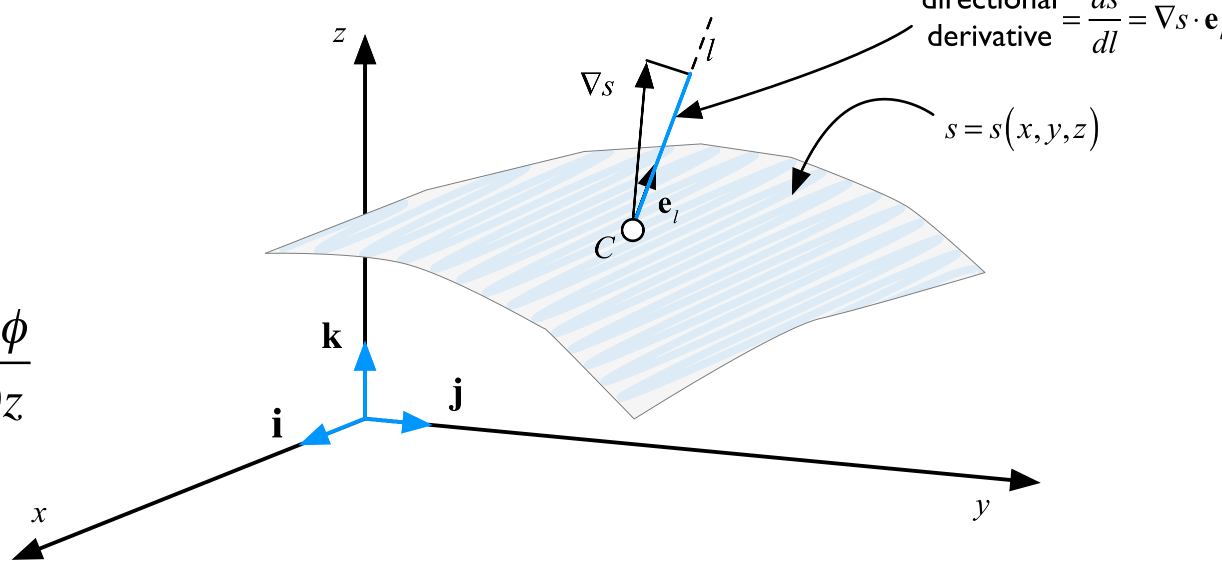
Gradients

Nabla operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Gradient

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$



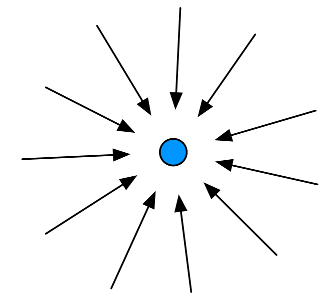
Directional Derivative

$$\frac{ds}{dl} = \nabla s \cdot \mathbf{e}_l = \|\nabla s\| \cos(\nabla s, \mathbf{e}_l)$$

Divergence

divergence

$$\nabla \cdot \mathbf{v} = \frac{\partial \mathrm{v}_x}{\partial x} + \frac{\partial \mathrm{v}_y}{\partial y} + \frac{\partial \mathrm{v}_z}{\partial z}$$

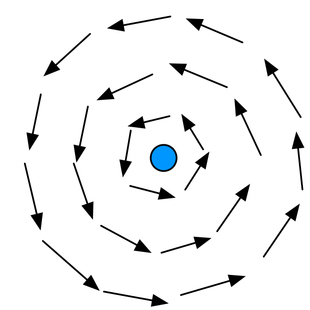


curl = 0

divergence of the gradient of a scalar is the Laplacian of the scalar

$$\nabla \cdot (\nabla s) = \nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$$

$$\Delta \phi = \nabla \cdot (\nabla \phi) = \nabla^2 \phi == \frac{\partial^2 \mathrm{v}_x}{\partial x^2} + \frac{\partial^2 \mathrm{v}_y}{\partial y^2} + \frac{\partial^2 \mathrm{v}_z}{\partial z^2}$$



divergence = 0

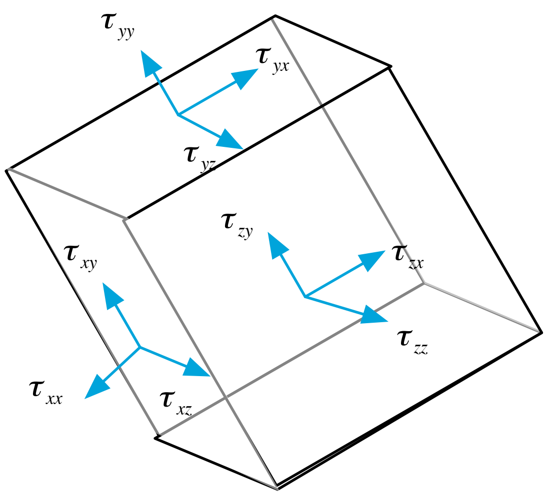
curl

$$\nabla \times \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \times (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k}$$

Tensors

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$



$$\boldsymbol{\tau} = \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \mathbf{ji}\tau_{yx} + \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz}$$

Other Tensors

$$\left.\begin{aligned}\{\mathbf{vv}\} &= (u\mathbf{i}+v\mathbf{j}+w\mathbf{k})(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\\ &= \mathbf{ii}uu+\mathbf{ij}uv+\mathbf{ik}uw+\\ &\quad \mathbf{ji}vu+\mathbf{jj}vv+\mathbf{jk}vw+\\ &\quad \mathbf{ki}wu+\mathbf{kj}wv+\mathbf{kk}ww\end{aligned}\right\} \Rightarrow \{\mathbf{vv}\} = \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix}$$

Velocity gradient

$$\left.\begin{aligned}\{\nabla\mathbf{v}\} &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\\ &= \mathbf{ii}\frac{\partial u}{\partial x}+\mathbf{ij}\frac{\partial v}{\partial x}+\mathbf{ik}\frac{\partial w}{\partial x}+\\ &\quad \mathbf{ji}\frac{\partial u}{\partial y}+\mathbf{jj}\frac{\partial v}{\partial y}+\mathbf{jk}\frac{\partial w}{\partial y}+\\ &\quad \mathbf{ki}\frac{\partial u}{\partial z}+\mathbf{kj}\frac{\partial v}{\partial z}+\mathbf{kk}\frac{\partial w}{\partial z}\end{aligned}\right\} \Rightarrow \{\nabla\mathbf{v}\} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Tensor Operations

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma} + \boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} + \tau_{xx} & \sigma_{xy} + \tau_{xy} & \sigma_{xz} + \tau_{xz} \\ \sigma_{yx} + \tau_{yx} & \sigma_{yy} + \tau_{yy} & \sigma_{yz} + \tau_{yz} \\ \sigma_{zx} + \tau_{zx} & \sigma_{zy} + \tau_{zy} & \sigma_{zz} + \tau_{zz} \end{bmatrix}$$

$$\left\{ s\boldsymbol{\tau} \right\} = \begin{bmatrix} s\tau_{xx} & s\tau_{xy} & s\tau_{xz} \\ s\tau_{yx} & s\tau_{yy} & s\tau_{yz} \\ s\tau_{zx} & s\tau_{zy} & s\tau_{zz} \end{bmatrix}$$

$$\left[\boldsymbol{\tau} \cdot \mathbf{v} \right] = \begin{pmatrix} \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \mathbf{ji}\tau_{yx} + \\ \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz} \end{pmatrix} \cdot \left(u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \right)$$

$$\left[\boldsymbol{\tau} \cdot \mathbf{v} \right] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \tau_{xx}u + \tau_{xy}v + \tau_{xz}w \\ \tau_{yx}u + \tau_{yy}v + \tau_{yz}w \\ \tau_{zx}u + \tau_{zy}v + \tau_{zz}w \end{bmatrix}$$

Tensor Operations

divergence of a tensor

$$\left[\nabla\cdot\boldsymbol{\tau}\right]=\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)\mathbf{i}+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)\mathbf{j}+\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)\mathbf{k}$$

double dot product of two tensors

$$\left(\boldsymbol{\tau}:\nabla\mathbf{v}\right)=\begin{pmatrix}\mathbf{ii}\tau_{xx}+\mathbf{ij}\tau_{xy}+\mathbf{ik}\tau_{xz}+\\ \mathbf{ji}\tau_{yx}+\mathbf{jj}\tau_{yy}+\mathbf{jk}\tau_{yz}+\\ \mathbf{ki}\tau_{zx}+\mathbf{kj}\tau_{zy}+\mathbf{kk}\tau_{zz}\end{pmatrix}:\begin{pmatrix}\mathbf{ii}\dfrac{\partial u}{\partial x}+\mathbf{ij}\dfrac{\partial v}{\partial x}+\mathbf{ik}\dfrac{\partial w}{\partial x}+\\ \mathbf{ji}\dfrac{\partial u}{\partial y}+\mathbf{jj}\dfrac{\partial v}{\partial y}+\mathbf{jk}\dfrac{\partial w}{\partial y}+\\ \mathbf{ki}\dfrac{\partial u}{\partial z}+\mathbf{kj}\dfrac{\partial v}{\partial z}+\mathbf{kk}\dfrac{\partial w}{\partial z}\end{pmatrix}$$

$$\left(\boldsymbol{\tau}:\nabla\mathbf{v}\right)=\tau_{xx}\frac{\partial u}{\partial x}+\tau_{xy}\frac{\partial u}{\partial y}+\tau_{xz}\frac{\partial u}{\partial z}+\tau_{yx}\frac{\partial v}{\partial x}+$$

$$\tau_{yy}\frac{\partial v}{\partial y}+\tau_{yz}\frac{\partial v}{\partial z}+\tau_{zx}\frac{\partial w}{\partial x}+\tau_{zy}\frac{\partial w}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}$$

Tensor Operations

Deformation tensor

$$\mathcal{D}(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

symmetric part of $\nabla\mathbf{v}$

Spin tensor

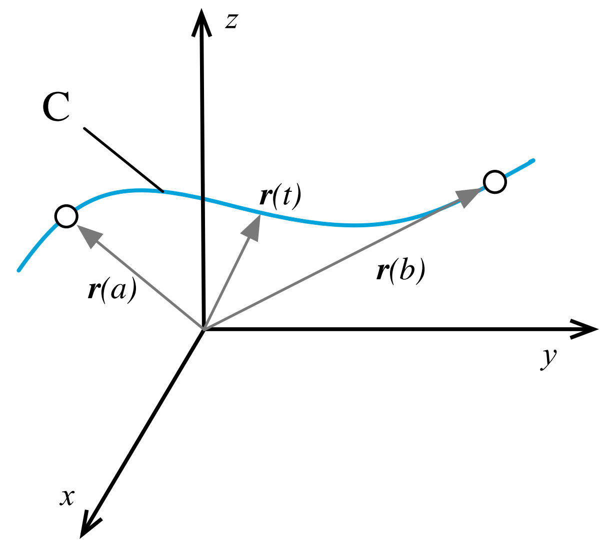
$$\mathcal{S}(\mathbf{v}) = \nabla\mathbf{v} - \mathcal{D}(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} - \nabla\mathbf{v}^T\right)$$

Skew-symmetric part of $\nabla\mathbf{v}$

$$trace(\nabla\mathbf{v}) = \nabla\cdot\mathbf{v}$$

Gradients Theorem for Line Integrals

$$\int_C \nabla s \cdot d\mathbf{r} = s\big(r(b)\big) - s\big(r(a)\big)$$

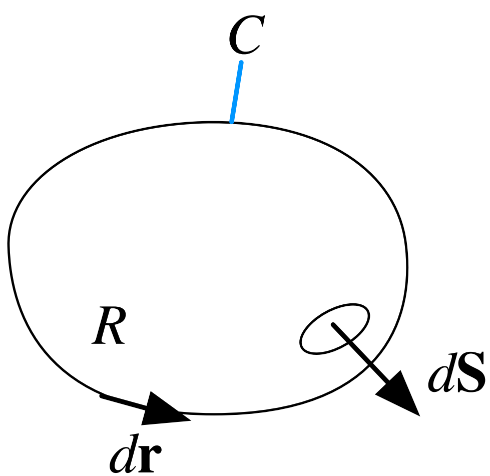


The gradient theorem for line integrals relates a line integral to the values of a function at its endpoints

Green's Theorem

Convert a volume integral to a surface integral and reduce the order of equations by one

$$\oint_C \left(udx + vdy\right) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

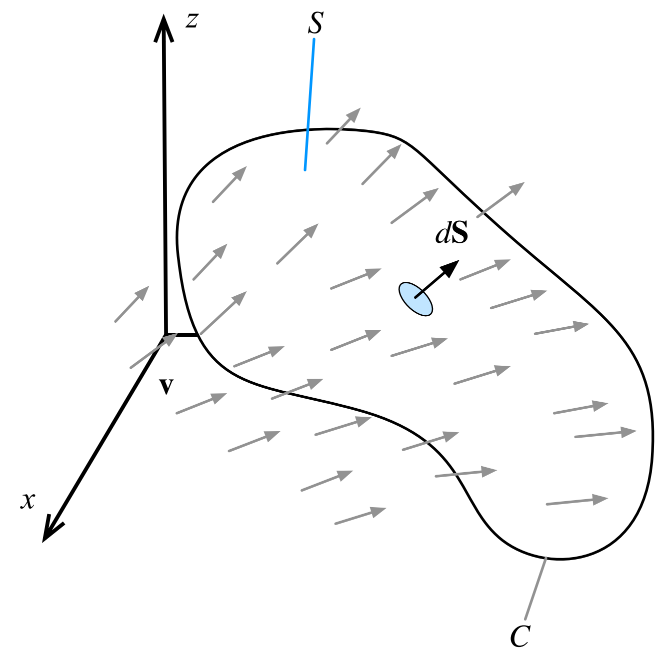


$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} \qquad \mathbf{v} = u\mathbf{i} + v\mathbf{j} \qquad d\mathbf{S} = dxdy\mathbf{k}$$

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_R \left[\nabla \times \mathbf{v}\right] \cdot d\mathbf{S}$$

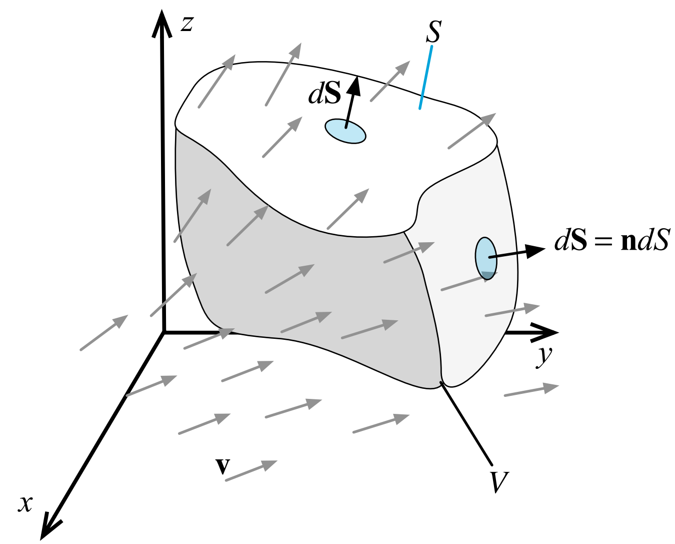
Stoke's Theorem

$$\int_S \left[\nabla \times \mathbf{v}\right] \cdot d\mathbf{S} = \oint_C \mathbf{v} \cdot d\mathbf{r}$$



Divergence Theorem

$$\int_V (\nabla \cdot \mathbf{v})\, dV = \oint_S \mathbf{v} \cdot \mathbf{n}\, dS$$



Leibniz Integral Rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} \phi(x,t)\,dx = \underbrace{\int_{a(t)}^{b(t)} \frac{\partial \phi}{\partial t}\,dx}_{Term\ I} + \underbrace{\phi\big(b(t),t\big)\frac{\partial b}{\partial t}}_{Term\ II} - \underbrace{\phi\big(a(t),t\big)\frac{\partial a}{\partial t}}_{Term\ III}$$