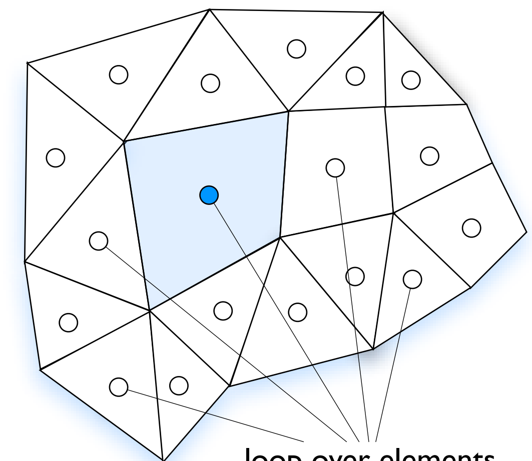
Treatment of Boundary Conditions

Advanced CFD 03

Momentum BCs

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \nabla\cdot\tau - \nabla p + \mathbf{B}$$

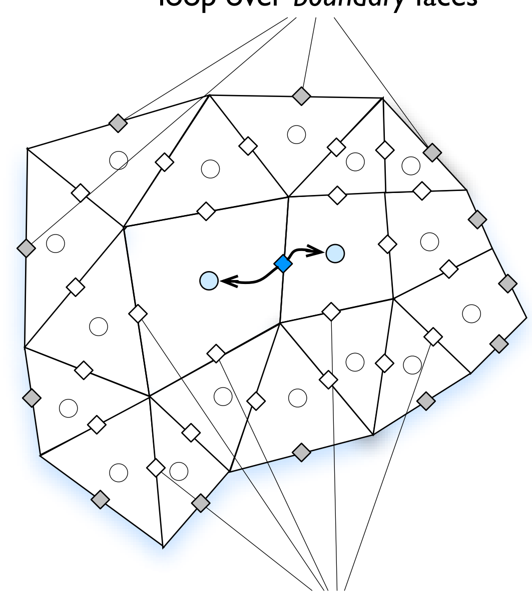


semi-discretized form

$$\frac{(\rho\mathbf{v})_P - (\rho\mathbf{v})_P^{\circ}}{\Delta t}\Omega_P + \sum_{f\sim nb(P)}\left(\dot{m}_f\mathbf{v}_f\right) = -\nabla P_P\Omega_P + \sum_{f\sim nb(P)}\left(\tau_f\cdot\mathbf{S}_f\right)$$

discretization type

$$\overbrace{\frac{(\rho\mathbf{v})_P - (\rho\mathbf{v})_P^{\circ}}{\Delta t}\Omega_P}^{\textit{element discretization}} + \underbrace{\sum_{f\sim nb(P)}\left(\dot{m}_f\mathbf{v}_f\right)}_{\textit{face discretization}} = \overbrace{-\nabla P_P\Omega_P}^{\textit{element discretization}} + \underbrace{\sum_{f\sim nb(P)}\left(\tau_f\cdot\mathbf{S}_f\right)}_{\textit{face discretization}}$$



boundary terms

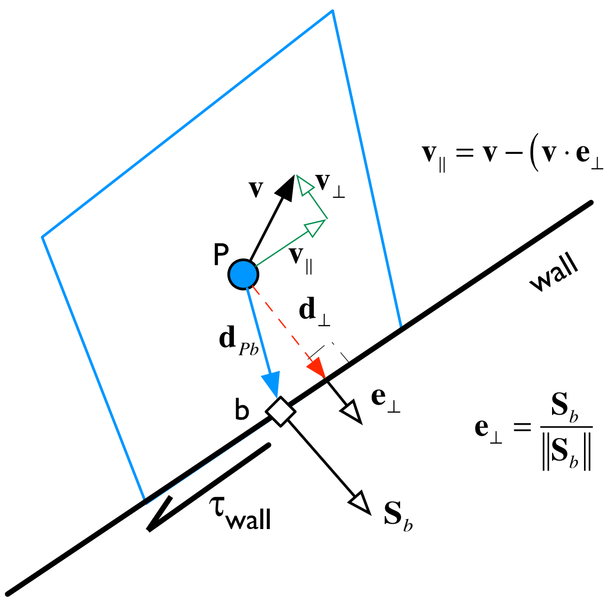
$$\underbrace{\sum_{f\sim nb(P)}\left(\dot{m}_f\mathbf{v}_f\right)}_{\textit{face discretization}} = \sum_{f\sim \text{int}\,erior\ nb(P)}\left(\dot{m}_f\mathbf{v}_f\right) + \underbrace{\dot{m}_b\mathbf{v}_b}_{\textit{boundary term}}$$

$$\underbrace{\sum_{f\sim nb(P)}\left(\tau_f\cdot\mathbf{S}_f\right)}_{\textit{face discretization}} = \sum_{f\sim \text{int}\,erior\ nb(P)}\left(\tau_f\cdot\mathbf{S}_f\right) + \underbrace{\tau_b\cdot\mathbf{S}_b}_{\textit{boundary term}} = \sum_{f\sim \text{int}\,erior\ nb(P)}\left(\tau_f\cdot\mathbf{S}_f\right) + \underbrace{\mathbf{F}_b}_{\textit{boundary term}}$$

No-Slip Wall

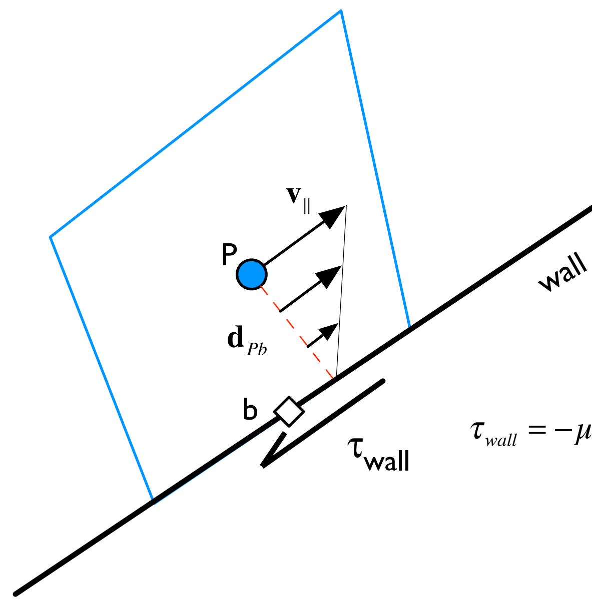
$$\mathbf{F}_b = \mathbf{F}_\perp + \mathbf{F}_\parallel$$
$$= \cancel{\mathbf{F}_\perp} + \mathbf{F}_\parallel$$

$$\mathbf{F}_\parallel = \tau_{wall} \left\| \mathbf{S}_b \right\|$$



$$\mathbf{v}_\parallel = \mathbf{v} - \left(\mathbf{v} \cdot \mathbf{e}_\perp\right)\mathbf{e}_\perp$$

$$\mathbf{e}_\perp = \frac{\mathbf{S}_b}{\left\| \mathbf{S}_b \right\|}$$



$$\tau_{wall} = -\mu \frac{\partial \mathbf{v}_\parallel}{\partial d_\perp}$$

$$\mathbf{v}_\parallel = \begin{Bmatrix} v_x - \left(v_x \cdot e_{\perp x} + v_y \cdot e_{\perp y}\right) e_{\perp x} \\ v_y - \left(v_x \cdot e_{\perp x} + v_y \cdot e_{\perp y}\right) e_{\perp y} \end{Bmatrix}$$

$$\mathbf{F}_b = -\frac{\mu \left\| \mathbf{S}_b \right\|}{d_\perp} \begin{Bmatrix} v_x - \left(v_x \cdot e_{\perp x} + v_y \cdot e_{\perp y}\right) e_{\perp x} \\ v_y - \left(v_x \cdot e_{\perp x} + v_y \cdot e_{\perp y}\right) e_{\perp y} \end{Bmatrix}$$

No-Slip Wall

treatment for x-component

$$\frac{(\rho v_x)_P - (\rho v_x)_P^{\circ}}{\Delta t}\Omega_P + \sum_{f\sim \text{int}\,erior\ nb(P)} \left(\dot{m}_f v_{x,f}\right) = -\left(\nabla P_P \cdot \mathbf{i}\right)\Omega_P + \sum_{f\sim \text{int}\,erior\ nb(P)} \left(\tau_f \cdot \mathbf{S}_f\right)\cdot \mathbf{i} - \frac{\mu\|\mathbf{S}_b\|}{d_\perp}\left(v_x - \left(v_x \cdot e_{\perp x} + v_y \cdot e_{\perp y}\right)e_{\perp x}\right)$$

$$-\frac{\mu\|\mathbf{S}_b\|}{d_\perp}\begin{Bmatrix} v_x\left(1-e_{\perp x}^2\right) - v_y \cdot e_{\perp y}e_{\perp x} \\ v_y\left(1-e_{\perp y}^2\right) - v_x \cdot e_{\perp x}e_{\perp y} \end{Bmatrix}$$

$$a_P \leftarrow a_P + \frac{\mu\|\mathbf{S}_b\|}{d_\perp}\left(1-e_{\perp x}^2\right)$$

$$b_P \leftarrow b_P + \frac{\mu\|\mathbf{S}_b\|}{d_\perp} v_y \cdot e_{\perp y}e_{\perp x}$$

similar treatment for y-component

Inlet

Outlet

Pressure BCs

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

semi-discretized form

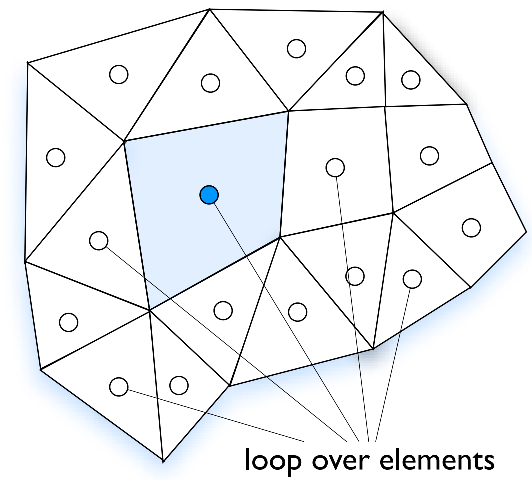
$$\frac{\rho - \rho^{\circ}}{\Delta t}\Omega + \sum_{f \sim nb(P)} (\rho \mathbf{v} \cdot \mathbf{S})_f = 0$$

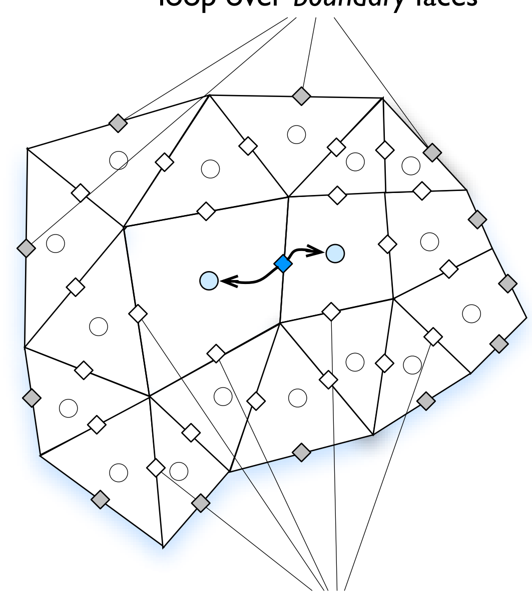
discretization type

$$\overbrace{\frac{\rho - \rho^{\circ}}{\Delta t}\Omega}^{element\ discretization} + \underbrace{\sum_{f \sim nb(P)} (\rho \mathbf{v} \cdot \mathbf{S})_f}_{face\ discretization} = 0$$

boundary terms

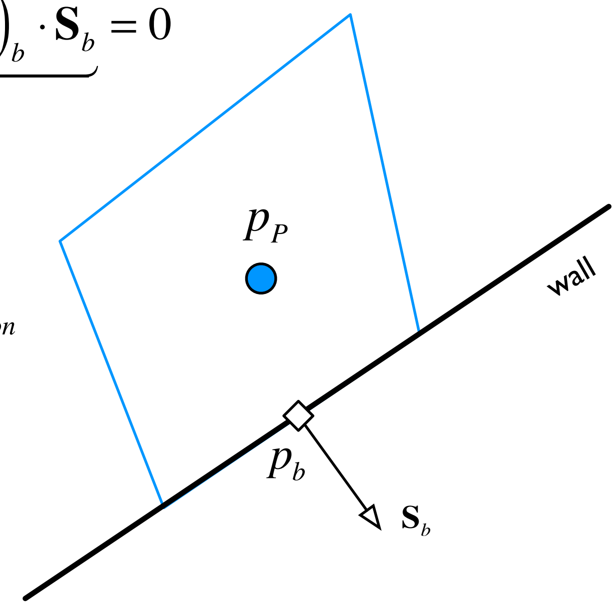
$$\frac{\rho - \rho^{\circ}}{\Delta t}\Omega + \sum_{f \sim \mathrm{int}\,erior\ nb(P)} (\rho \mathbf{v} \cdot \mathbf{S})_f + \underbrace{(\rho \mathbf{v} \cdot \mathbf{S})_b}_{boundary\ term} = 0$$





Boundary Terms Contribution

$$\frac{\rho^* + \rho' - \rho^\circ}{\Delta t}\Omega + \sum_{f\sim \text{int}\,erior\ nb(P)} \left(\rho^*+\rho'\right)_f \left(\mathbf{v}^*+\mathbf{v}'\right)_f \cdot \mathbf{S}_f + \underbrace{\left(\rho^*+\rho'\right)_b\left(\mathbf{v}^*+\mathbf{v}'\right)_b\cdot\mathbf{S}_b}_{boundary\ term} = 0$$



$$a_p = \underbrace{\frac{V_P C_\rho}{\Delta t}}_{elements\ contribution} + \underbrace{\sum_{f=nb(P)} \left(\frac{C_{\rho,f}}{\rho_f^{(n)}}\left\|\dot{m}_f^*,0\right\|\right) + \sum_{f=nb(P)} \rho_f^{(n)}\mathcal{D}_f}_{\text{int}\,erior\ \ faces\ \ contribution} + \underbrace{?}_{boundary\ \ face\ \ contribution}$$

$$a_F = \underbrace{-\left\|-\dot{m}_f^*,0\right\|\frac{C_{\rho,f}}{\rho_f^{(n)}} - \rho_f^{(n)}\mathcal{D}_f}_{\text{int}\,erior\ \ faces\ \ contribution}$$

$$b_P = \underbrace{-\frac{\left(\rho_P^{(n)}-\rho_P^\circ\right)}{\Delta t}V}_{elements\ contribution\ P} + \underbrace{\sum_{f=nb(P)} -\dot{m}_f^* + \sum_{f=nb(P)} \rho_f^{(n)}\left(\mathbf{D}_f \nabla p'_f\right)\cdot \mathbf{T}_f}_{\text{int}\,erior\ \ faces\ \ contribution} + \underbrace{?}_{boundary\ \ face\ \ contribution}$$

No-Slip Wall

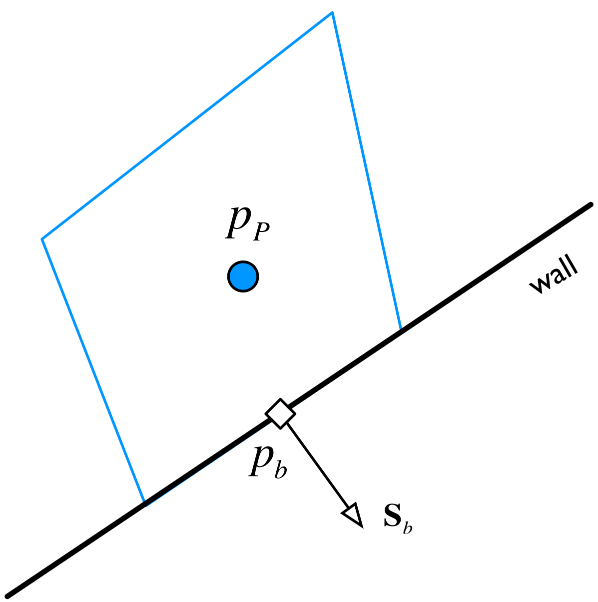
$$\left(\rho \mathbf{v} \cdot \mathbf{S}\right)_b = 0$$

boundary contribution = 0

$$a_p = \underbrace{\frac{V_P C_\rho}{\Delta t}}_{elements\ contribution} + \underbrace{\sum_{f=nb(P)} \left(\frac{C_{\rho,f}}{\rho_f^{(n)}} \left\| \dot{m}_f^*, 0 \right\| \right) + \sum_{f=nb(P)} \rho_f^{(n)} \mathcal{D}_f}_{\text{int}\,erior\ \ faces\ \ contribution}$$

$$a_F = \underbrace{-\left\| -\dot{m}_f^*, 0 \right\| \frac{C_{\rho,f}}{\rho_f^{(n)}} - \rho_f^{(n)} \mathcal{D}_f}_{\text{int}\,erior\ \ faces\ \ contribution}$$

$$b_P = -\underbrace{\frac{\left(\rho_P^{(n)} - \rho_P^{\circ}\right)}{\Delta t} V}_{elements\ contribution\ P} + \underbrace{\sum_{f=nb(P)} -\dot{m}_f^* + \sum_{f=nb(P)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p_f'\right) \cdot \mathbf{T}_f}_{\text{int}\,erior\ \ faces\ \ contribution}$$



Boundary Vallues

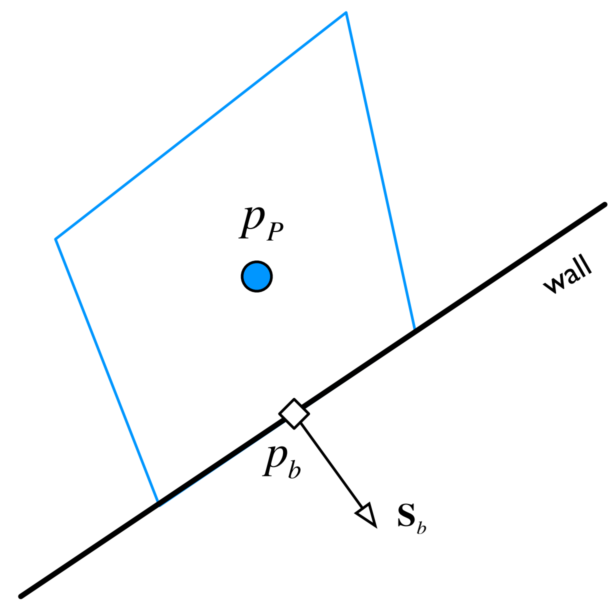
using Rhie-Chow interpolation

$$(\rho \mathbf{v} \cdot \mathbf{S})_b = 0 \qquad \Rightarrow \rho_b^{(n)} \overline{\mathbf{v}_b} \cdot \mathbf{S}_b - \rho_b^{(n)} \overline{\mathbf{D}_b} \left(\nabla p_b - \overline{\nabla p_b} \right) \cdot \mathbf{S}_b = 0$$

$$\overline{\square}_b = \square_P \qquad \Rightarrow \rho_b^{(n)} \mathbf{v}_P \cdot \mathbf{S}_b - \rho_b^{(n)} \mathbf{D}_P \left(\nabla p_b - \nabla p_P \right) \cdot \mathbf{S}_b = 0$$

$$\Rightarrow \mathbf{D}_P \left(\nabla p_b^{(n)} - \nabla p_P^{(n)} \right) \cdot \mathbf{S}_b = 0$$

$$\Rightarrow \quad p_b = p_P + \frac{\left(\mathbf{D}_P \nabla p_P^{(n)} \right) \cdot \mathbf{S}_b - \left(\mathbf{D}_P \nabla p_b^{(n-1)} \right) \cdot \mathbf{T}_b}{\mathcal{D}_P}$$



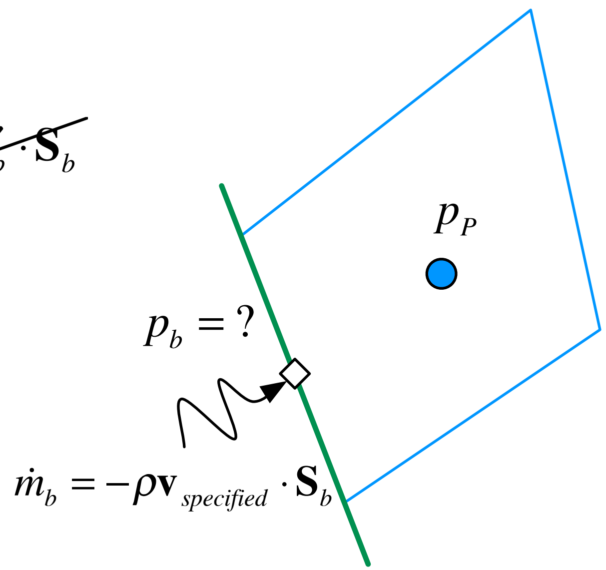
assuming a profile

$$\Rightarrow \qquad p_b = \begin{cases} p_P & constant \quad profile \\ p_P + \nabla p_P \cdot \mathbf{d}_{Pb} & linear \quad profile \end{cases}$$

Subsonic Inlet Specified Velocity

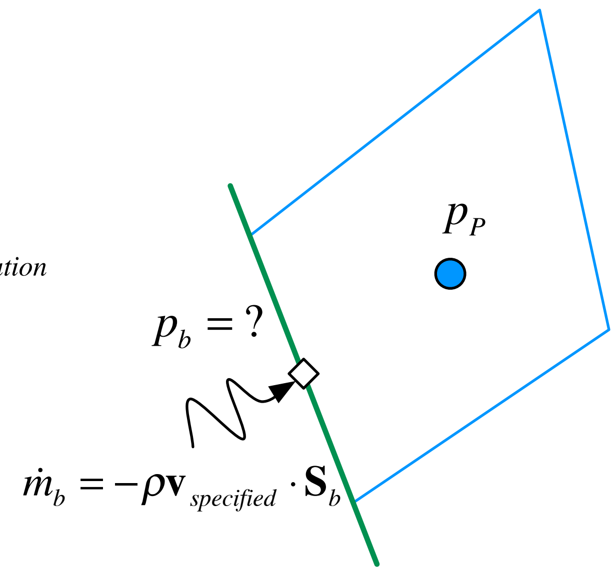
$$\underbrace{\left(\rho^* + \rho'\right)_b \left(\mathbf{v}^* + \mathbf{v}'\right)_b \cdot \mathbf{S}_b}_{boundary\ term} = \rho_b^* \mathbf{v}_b^* \cdot \mathbf{S}_b + \cancel{\rho_b^* \mathbf{v}_b' \cdot \mathbf{S}_b} + \rho_b' \mathbf{v}_b^* \cdot \mathbf{S}_b + \cancel{\rho_b' \mathbf{v}_b' \cdot \mathbf{S}_b}$$

$$== \dot{m}_b^* + \frac{\dot{m}_b^*}{\rho_b^*} \rho_b'$$



Contributions

$$a_p = \frac{V_P C_\rho}{\Delta t} + \underbrace{\sum_{f=nb(P)} \left(\frac{C_{\rho,f}}{\rho_f^{(n)}} \left\| \dot{m}_f^*, 0 \right\| \right) + \sum_{f=nb(P)} \rho_f^{(n)} \mathrm{D}_f}_{interior\ \ faces\ \ contribution} + \underbrace{C_{\rho,b} \frac{\dot{m}_b^*}{\rho_b^{(n)}}}_{boundary\ \ face\ \ contribution}$$



$$b_P = \underbrace{-\frac{\left(\rho_P^{(n)} - \rho_P^\circ\right)}{\Delta t} V}_{elements\ contribution_P} + \underbrace{\sum_{f=nb(P)} -\dot{m}_f^* + \sum_{f=nb(P)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p'_f \right) \cdot \mathbf{T}_f}_{interior\ \ faces\ \ contribution} + \underbrace{\dot{m}_b}_{boundary\ \ face\ contribution}$$

Specified Velocity

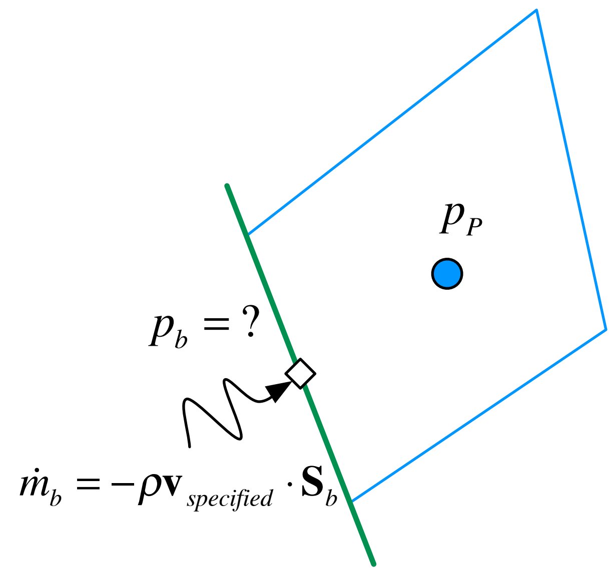
using Rhie-Chow interpolation

$$\dot{m}_b = \rho_b^{(n)} \overline{\mathbf{v}_b} \cdot \mathbf{S}_b - \rho_b^{(n)} \overline{\mathbf{D}_b} \left(\nabla p_b - \overline{\nabla p_b} \right) \cdot \mathbf{S}_b$$

$$\overline{\square}_b = \square_P \qquad \dot{m}_b = \rho_b^{(n)} \mathbf{v}_P \cdot \mathbf{S}_b + \rho_b^{(n)} \mathbf{D}_P \left(\nabla p_b - \nabla p_P \right) \cdot \mathbf{S}_b$$

$$\Rightarrow \mathbf{D}_P \left(\nabla p_b^{(n)} - \nabla p_P^{(n)} \right) \cdot \mathbf{S}_b = \frac{\dot{m}_b}{\rho_b^{(n)}} - \mathbf{v}_P \cdot \mathbf{S}_b$$

$$\Rightarrow \quad p_b = p_P + \frac{\dfrac{\dot{m}_b}{\rho_b^{(n)}} - \mathbf{v}_P \cdot \mathbf{S}_b + \left(\mathbf{D}_P \nabla p_P^{(n)} \right) \cdot \mathbf{S}_b - \left(\mathbf{D}_P \nabla p_b^{(n-1)} \right) \cdot \mathbf{T}_b}{\mathcal{D}_P}$$



assuming a profile

$$\Longrightarrow \qquad p_b = \begin{cases} p_P & \textit{constant profile} \\ p_P + \nabla p_P \cdot \mathbf{d}_{Pb} & \textit{linear profile} \end{cases}$$

Subsonic Inlet Specified Pressure

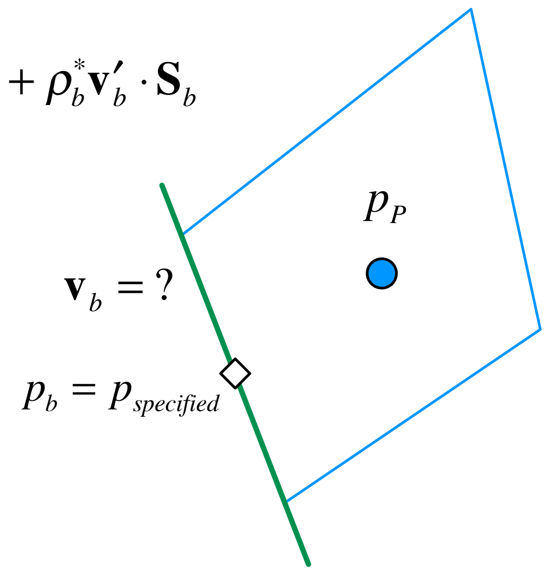
$$\underbrace{\left(\rho^* + \rho'\right)_b \left(\mathbf{v}^* + \mathbf{v}'\right)_b \cdot \mathbf{S}_b}_{boundary\ term} = \rho_b^* \mathbf{v}_b^* \cdot \mathbf{S}_b + \rho_b^* \mathbf{v}_b' \cdot \mathbf{S}_b + \cancel{\rho_b' \mathbf{v}_b^* \cdot \mathbf{S}_b} + \cancel{\rho_b' \mathbf{v}_b' \cdot \mathbf{S}_b} = \dot{m}_b^* + \rho_b^* \mathbf{v}_b' \cdot \mathbf{S}_b$$

$$\rho_b^* \mathbf{v}_b' \cdot \mathbf{S}_b = \rho_b^* \cancel{\overline{\mathbf{v}}_b} \cdot \mathbf{S}_b - \rho_b^{(n)} \overline{\mathbf{D}}_b \left(\nabla p_b' - \overline{\nabla p_b'}\right) \cdot \mathbf{S}_b$$

$$\overline{\square}_b = \square_P \qquad = -\rho_b^{(n)} \mathbf{D}_P \left(\nabla p_b' - \nabla p_P'\right) \cdot \mathbf{S}_b$$

$$= -\rho_b^{(n)} \mathcal{D}_P \left(p_b' - p_P'\right) - \mathbf{D}_P \left(\nabla p_b' \cdot \mathbf{T}_b - \nabla p_P' \cdot \mathbf{S}_b\right)$$

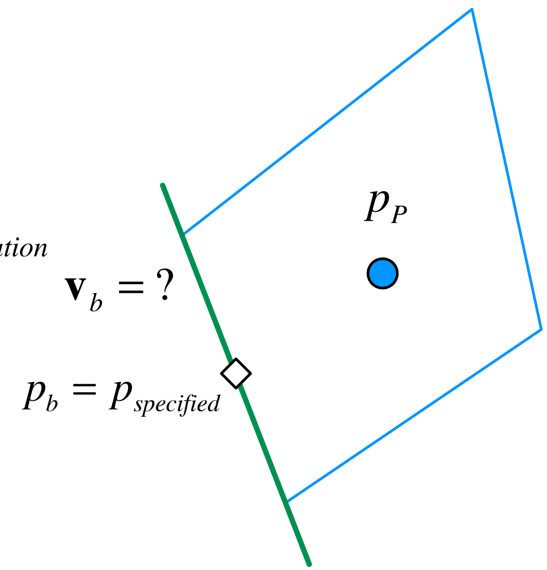
$$p_b' = 0$$



Contributions

$$a_p = \frac{V_P C_\rho}{\Delta t} + \underbrace{\sum_{f=nb(P)} \left(\frac{C_{\rho,f}}{\rho_f^{(n)}} \left\| \dot{m}_f^*, 0 \right\| \right) + \sum_{f=nb(P)} \rho_f^{(n)} \mathcal{D}_f}_{\text{interior faces contribution}} + \underbrace{\rho_b^{(n)} \mathcal{D}_P}_{\textit{boundary face contribution}}$$

$$a_F = \underbrace{-\left\| -\dot{m}_f^*, 0 \right\| \frac{C_{\rho,f}}{\rho_f^{(n)}} - \rho_f^{(n)} \mathcal{D}_f}_{\text{interior faces contribution}}$$

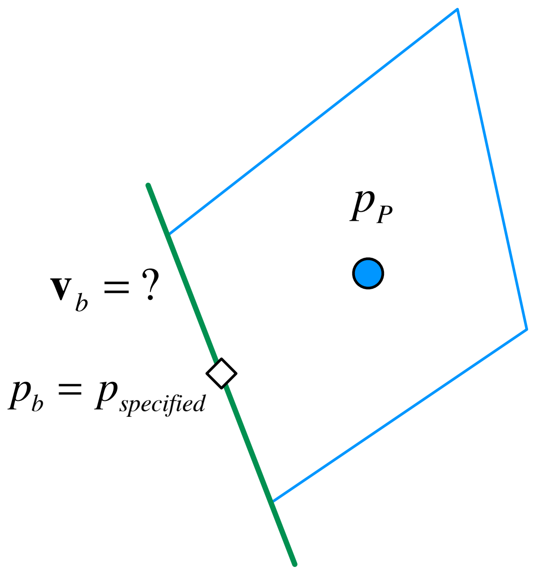


$$b_P = \underbrace{-\frac{\left(\rho_P^{(n)} - \rho_P^{\circ} \right)}{\Delta t} V}_{\textit{elements contribution } P} + \underbrace{\sum_{f=nb(P)} -\dot{m}_f^* + \sum_{f=nb(P)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p'_f \right) \cdot \mathbf{T}_f}_{\text{interior faces contribution}} + \underbrace{\dot{m}_b + \mathbf{D}_P \left(\nabla p'_b \cdot \mathbf{T}_b - \nabla p'_P \cdot \mathbf{S}_b \right)}_{\textit{boundary face contribution}}$$

Boundary Values

$$\dot{m}_b = \rho_b^{(n)} \mathbf{v}_P^* \cdot \mathbf{S}_b - \rho_b^{(n)} \mathbf{D}_P \left(\nabla p_b^{(n)} - \nabla p_P^{(n)} \right) \cdot \mathbf{S}_b$$

$$\Rightarrow v_b^* = \frac{\dot{m}_b}{\left(\mathbf{e}_{\mathbf{v}_b} \cdot \mathbf{S}_b \right) \rho_b^{(n)}}$$



Subsonic Outlet

$$\dot{m}_b^* = \rho_b^{(n)} \mathbf{v}_P^* \cdot \mathbf{S}_b - \rho_b^{(n)} \mathbf{D}_P \left(\nabla p_b^{(n)} - \nabla p_P^{(n)} \right) \cdot \mathbf{S}_b$$

$$a_p = \begin{cases} \dfrac{V_P C_\rho}{\Delta t} + \underbrace{\sum_{f=nb(P)} \rho_f^{(n)} \mathsf{D}_f}_{\text{int}\,erior\ faces\,contribution} + \underbrace{\rho_b^{(n)} \mathsf{D}_P}_{boundary\ face\,contribution} & incompressible \\ \dfrac{V_P C_\rho}{\Delta t} + \underbrace{\sum_{f=nb(P)} \left(\dfrac{C_{\rho,f}}{\rho_f^{(n)}} \left\| \dot{m}_f^*, 0 \right\| \right) + \sum_{f=nb(P)} \rho_f^{(n)} \mathsf{D}_f}_{\text{int}\,erior\ faces\,contribution} + \underbrace{\dfrac{C_{\rho,b}}{\rho_b^{(n)}} \dot{m}_b^* + \rho_b^{(n)} \mathsf{D}_P}_{boundary\ face\,contribution} & compressible \end{cases}$$

Subsonic Outlet Specified Pressure

$$\frac{\rho^* + \rho' - \rho^\circ}{\Delta t}\Omega + \sum_{f\sim \mathrm{int}\,erior\ nb(P)} \left(\rho^* + \rho'\right)_f \left(\mathbf{v}^* + \mathbf{v}'\right)_f \cdot \mathbf{S}_f + \underbrace{\left(\rho^* + \rho'\right)_b \left(\mathbf{v}^* + \mathbf{v}'\right)_b \cdot \mathbf{S}_b}_{boundary\ term} = 0$$

$$P'_b = 0 \qquad \Rightarrow \rho'_b = 0$$

$$\overline{\square}_b = \square_P \qquad \mathbf{v}'_b = \cancel{\mathbf{v}'_P} - \mathbf{D}_P\left(\nabla P'_b - \nabla P'_P\right)$$

$$\begin{aligned}
\mathbf{v}'_b \cdot \mathbf{S}_b &= -\mathbf{D}_P\left(\nabla P'_b - \nabla P'_P\right)\cdot \mathbf{S}_b \\
&= D_P \frac{\cancel{P'_b} - P'_P}{d_{Pb}}\mathbf{E}_b - \mathbf{D}_P \nabla P'_b \cdot \left(\mathbf{S}_b - \mathbf{E}_b\right) + \nabla P'_P \cdot \mathbf{S}_b
\end{aligned}$$

$$\underbrace{\left(\rho^* + \rho'\right)_b \left(\mathbf{v}^* + \mathbf{v}'\right)_b \cdot \mathbf{S}_b}_{boundary\ term} = \rho^*_b \mathbf{v}^*_b \cdot \mathbf{S}_b + \rho^*_b \mathbf{v}'_b \cdot \mathbf{S}_b = \dot{m}^*_b + \rho^*_b \mathbf{v}'_b \cdot \mathbf{S}_b$$

$$-\rho^*_b D_P \frac{\cancel{P'_b} - P'_P}{d_{Pb}}\mathbf{E}_b - \cancel{\rho^*_b \mathbf{D}_P \nabla P'_b \cdot \left(\mathbf{S}_b - \mathbf{E}_b\right)} + \rho^*_b \nabla P'_P \cdot \mathbf{S}_b$$