Simulating bi-Dimensional Turbulence in Fusion Plasmas

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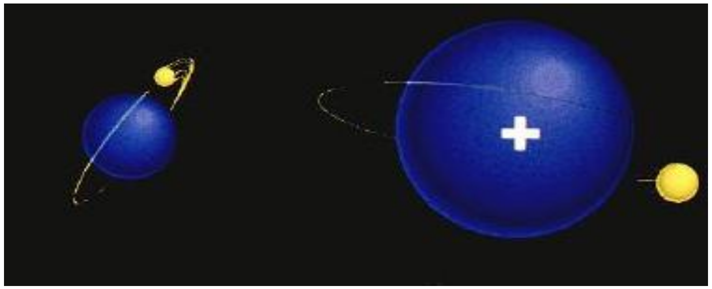
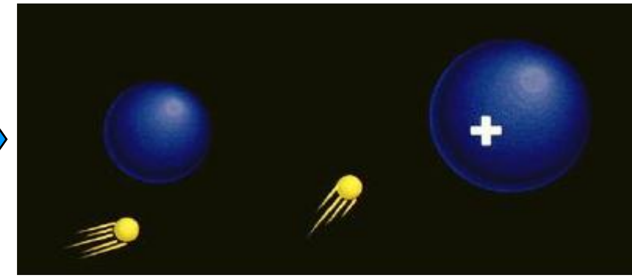
Outline

1. Fusion Plasmas & Magnetic Confinement
2. Turbulence in Plasma Devices
3. The Hasegawa-Mima Model
4. Mathematical Formulation
5. Discretization of the Equation
6. Tests and Results
7. Conclusion and Future work

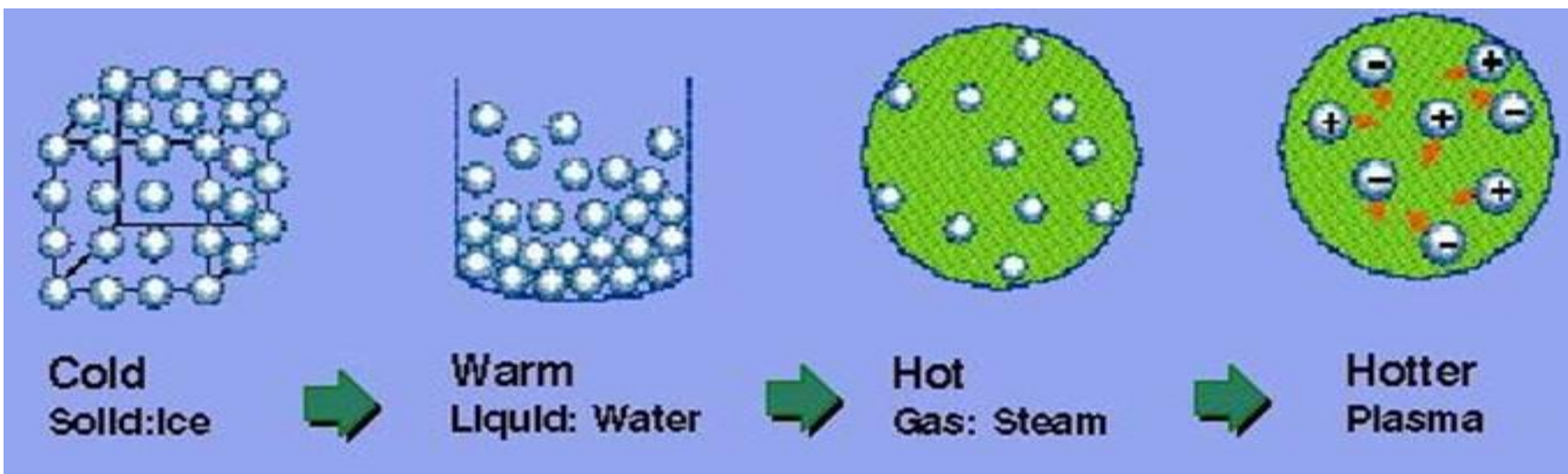
A plasma is an ionized gas



+ Energy (like heat) →



- Plasmas are the most common form of matter, comprising more than 99% of the visible universe.
- Plasmas carry electrical currents and generate magnetic fields, due to their ions and electrons.



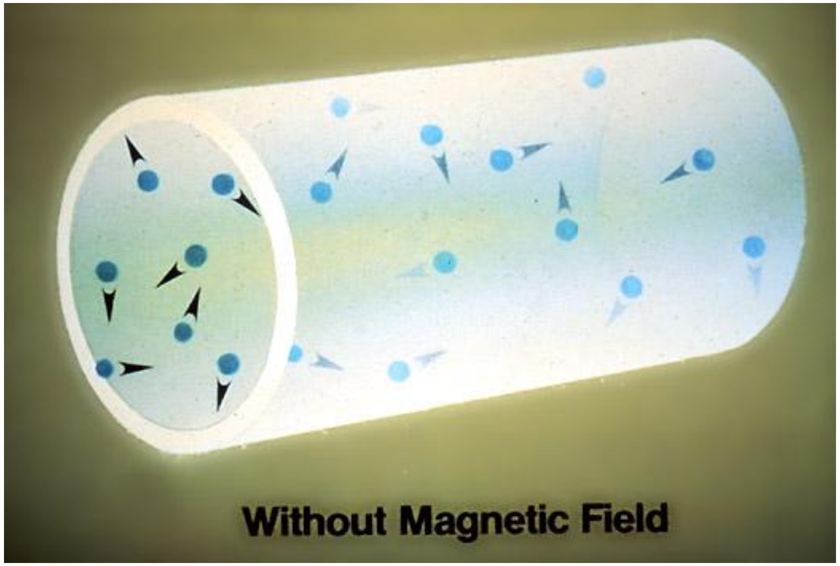
Main plasma properties:

1. Plasma is formed of free electrons and ions
2. The mass of the electrons is 2000 times smaller than the mass of the ions i.e. $m_e \ll m_i$
3. Each of the species (e^- and i^+) has:

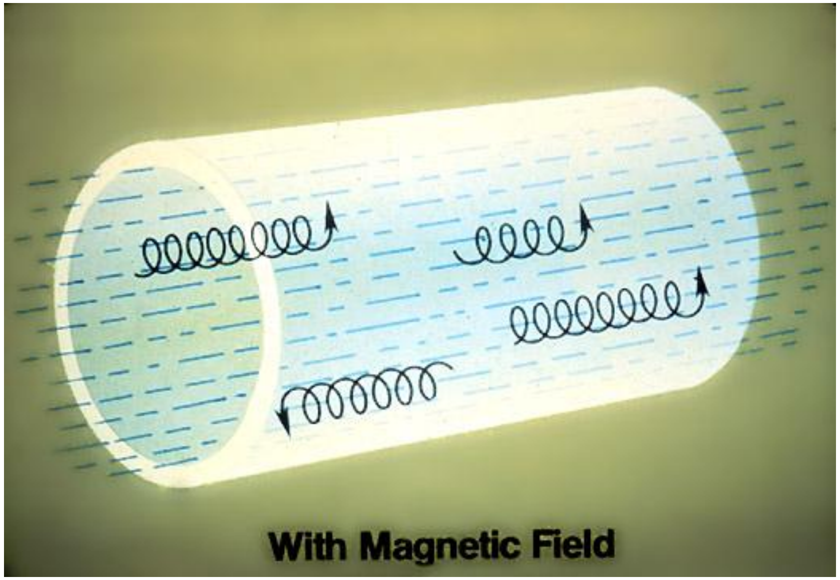
   density, temperature, pressure and velocity
4. Most plasmas are quasi-neutral: $n_e \approx n_i$
5. A plasma in which the magnetic field is strong enough to influence the motion of the charged particles is said to be magnetized.

The need for a magnetic field to confine the plasma

Plasma without a magnetic field: Particles get away from each other due to their thermal velocity.



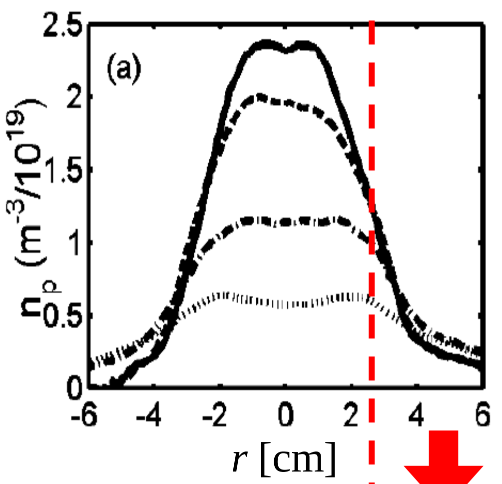
Adding a magnetic field with parallel set of coils: Particles trajectories tend to be parallel to **B**.

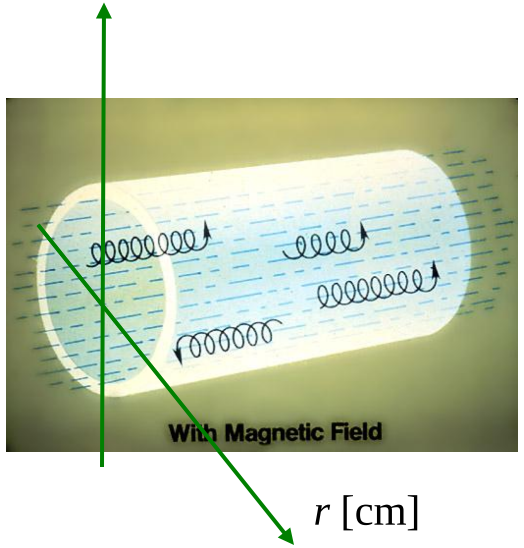


Turbulence in Plasma Devices

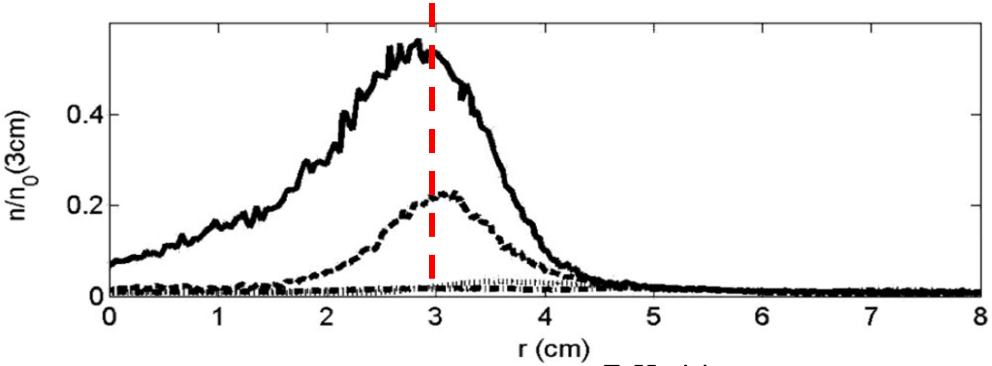
Experiments show that density gradient causes the onset of instabilities & turbulence perpendicular to **B**

The plasma density increases with the magnetic field
⇨ Leading to steeper density gradients

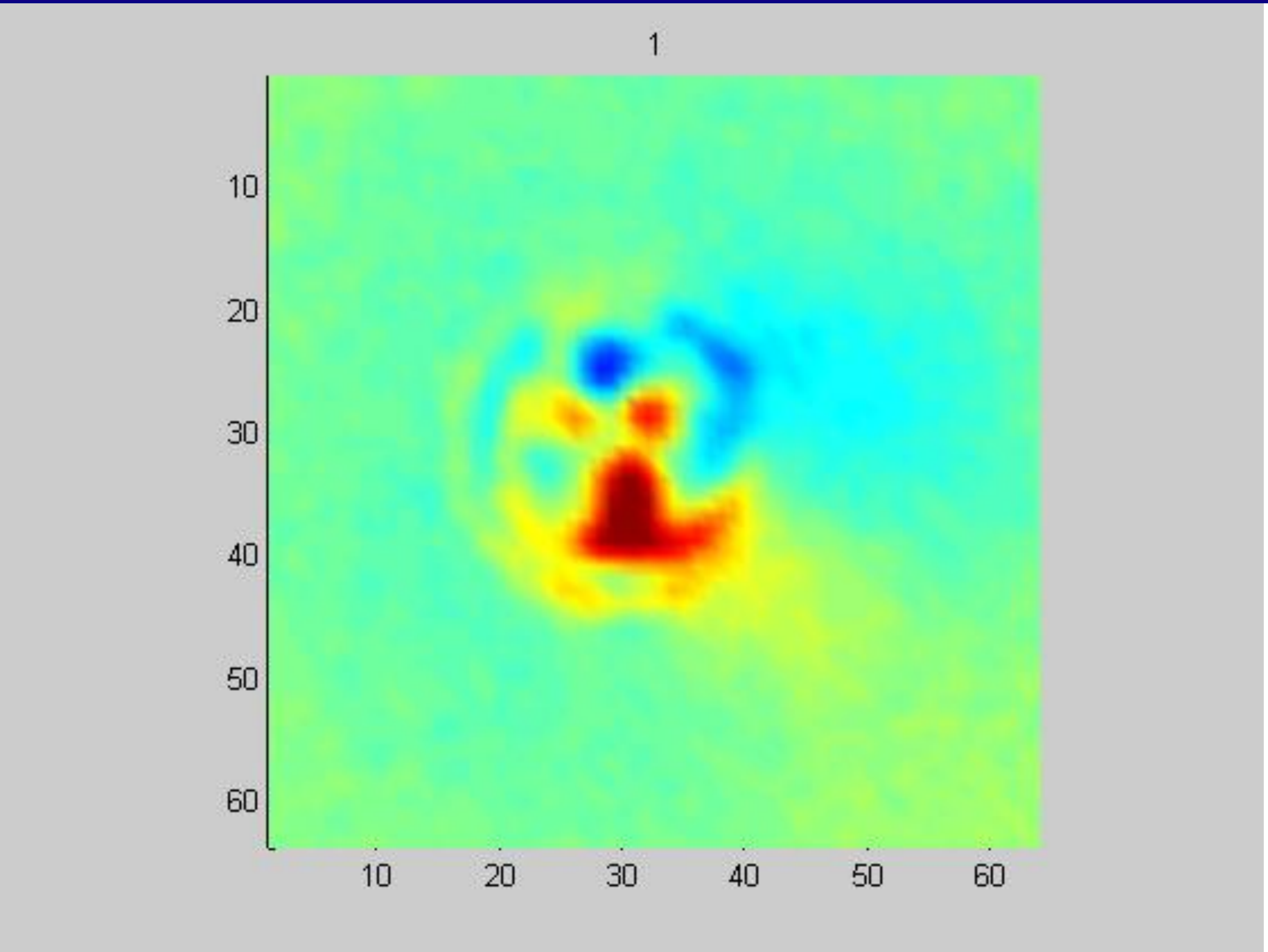


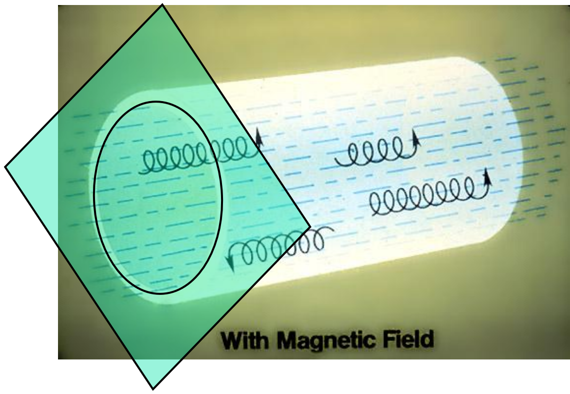


-The level of fluctuations increases above a certain critical gradient, and the plasma becomes turbulent
- Fluctuations increase where the gradient is steepest



Imaging density fluctuations in the cylindrical plasma





Camera settings:
Integration time 1 μs
Time between frames 15 μs
32x32 pixels

The Hasegawa Mima Model

Navier-Stokes Equations

The equations used are the Navier-Stokes equations for e^- and i^+

1- Conservation of the number of electrons or ions : $\dfrac{\partial n_\alpha}{\partial t} + \vec{\nabla}(n_\alpha \vec{u}_\alpha) = 0$

2- Conservation of momentum : $m_\alpha n_\alpha \left[\dfrac{\partial \vec{u}_\alpha}{\partial t} + (\vec{u}_\alpha . \vec{\nabla}) \vec{u}_\alpha \right] = -\nabla p_\alpha + q_\alpha n_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B})$

Particle balance

- 1 eq for electrons
- 1 eq for ions

= 2 equations

Momentum balance

- 2 eqs for the // direction to B (1 for the e- and 1 for the i+)
- 2 eqs for the $\perp$ direction to B (1 for the e- and 1 for the i+)

= 4 equations

= 6 equations in total

Properties of the Hasegawa-Mima model

- Generalised energy and Enstrophy conserved
- Contains a specific scale length, i.e. ρ_s
- For $k\rho_s >> 1$ similar to 2D incompressible Euler fluid.
- Homogeneous background magnetic field

$$\mathbf{B} = B_0\hat{z}$$

- Inhomogeneous plasma

$$n_0 = n_0(x)$$

- Cold ions (ion thermal balance equation dropped) and

$$T_i << T_e$$

Derivation of the Hasegawa-Mima Equation

- Start with:

  ⇒ Ion continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

  ⇒ Ion momentum balance equation

$$m_i n \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p_i + en(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

  ⇒ Electrostatic perturbations $\mathbf{E} = -\nabla\phi$

  ⇒ Cold ions $\nabla p \to 0$ $p = nT$ and $T_i \to 0$

- The momentum balance equation in the parallel direction leads to the **adiabatic relationship**

$$\frac{\tilde{n}}{n_0} = \frac{e\phi}{T_e}$$

The velocity in the direction perpendicular to $\mathbf{B}$

Using **perturbation theory**, we obtain the relationship between the velocity and the electrostatic potential ϕ

$$-\frac{e}{m}\nabla\phi + \frac{e}{m}\mathbf{u}\times\mathbf{B} = 0 \Rightarrow \mathbf{u} = \mathbf{u}_E = -\nabla\phi\times\frac{\mathbf{B}}{B_0^2}$$

ExB drift

- Substituting $\mathbf{u}_E$ in the momentum balance, one has a next order correction

$$\frac{\partial}{\partial t}\mathbf{u}_E + (\mathbf{u}_E\cdot\nabla)\mathbf{u}_E = \frac{e}{m}\mathbf{u}\times\mathbf{B} \Rightarrow \mathbf{u} = \mathbf{u}_P = \frac{1}{\omega_{ci}B_0}\left[-\frac{\partial}{\partial t}\nabla_\perp\phi - (\mathbf{u}_E\cdot\nabla_\perp)\nabla_\perp\phi\right]$$

Polarisation drift

- Thus the total velocity is

$$\boxed{\mathbf{u} = -\nabla\phi\times\frac{\mathbf{B}}{B_0^2} + \frac{1}{\omega_{ci}B_0}\left[-\frac{\partial}{\partial t}\nabla_\perp\phi - (\mathbf{u}_E\cdot\nabla_\perp)\nabla_\perp\phi\right]}$$

Using the continuity equation with the relationships found of the velocities and the density with the electrostatic potential

Using the continuity equation and adiabaticity

$$\frac{dlnn}{dt} + \nabla \cdot \mathbf{u} = 0 \qquad \frac{\tilde{n}}{n_0} = \frac{e\phi}{T_e}$$

- Substituting $\mathbf{u}_P$, we get

$$\frac{d}{dt}\left(\frac{1}{\omega_{ci}B_0}\nabla_\perp^2\phi - \frac{e\phi}{T_e}\right) + (\mathbf{u}_E \cdot \nabla) ln\frac{n_0}{\omega_{ci}} = 0$$

- Use normalisation

$$\omega_{ci}t \to t \quad \frac{x,y}{\rho_s} \to x,y \quad \frac{e\phi}{T_e} \to \phi \;\; where \;\; \rho_s = \sqrt{\frac{T_e}{m_i}} \cdot \frac{1}{\omega_{ci}}$$

To get

$$\frac{\partial}{\partial t}(\nabla^2\phi - \phi) - [(\nabla\phi \times \hat{z}) \cdot \nabla]\left[\nabla^2\phi - ln\left(\frac{n_0}{\omega_{ci}}\right)\right] = 0$$

Hasegawa-Mima equation

Mathematical Formulation

Mathematical Formulation

Developing the Hasegawa-Mima equation derived in the previous section, and using the Poisson brackets, leads to a new expression of the equation, that is:

$$\partial_t(\Delta\phi - \phi) - \{\phi, \Delta\phi + ln(\frac{n_0}{\omega ci})\} = 0$$

Let $J(\phi) = -\{\phi, \Delta\phi + ln(n_0/\omega_{ci})\}$,

$$(\mathrm{P}) \quad \begin{cases} \partial_t(-\Delta\phi + \phi) = J(\phi) \ \text{ in } \Omega \\ \phi \text{ dpc on } \Gamma = \partial\Omega \\ \phi(x, y, 0) = \phi_0(x, y) \end{cases}$$

We introduce now a variational formulation for (P). For that purpose let

$$\xi = \{v \in H^1(\Omega) | v(0, y) = v(L, y), \forall y \in (0, L) \text{ and } v(x, 0) = v(x, L) \ \forall x \in (0, L)\}$$

Mathematical Formulation

where $$H^1(\Omega) = \{v \in L^2(\Omega) | v_x, v_y \in L^2(\Omega)\}$$

is the usual Sobolev space of order 1, the derivatives being taken in L^2.

To provide a weak formulation for (P), we start by considering the elliptic problem

$$\text{(P')} \qquad \begin{cases} u : \bar{\Omega} \longrightarrow \mathbb{R} \\ -\Delta u + u = f \ \text{ in } \Omega \ \ (f \in L^2(\Omega)) \\ u \text{ dpc on } \Gamma \end{cases}$$

Let $\quad a(u,v) = \int_\Omega (\nabla u \nabla v + uv)\, dxdy \quad$ and $\quad < f, v >= \int_\Omega fv\, dxdy.$

Then (P') can be formulated as follows:

$$(\mathrm{P}_\alpha) \qquad u \in \xi : \ a(u,v) =< f, v >, \ \ \forall v \in \xi$$

Mathematical Formulation

Using Lax-Milgram theorem, problem (P_α) has a unique solution $u \in \xi$ verifying (P_α). Define the map:

$$S : f \longrightarrow u = S(f),$$

where S is the solution operator of (P_α).

On that basis, we reformulate (P), whereas one seeks for $T > 0$,

$$\phi : [0, T] \longrightarrow \xi$$

$$\frac{d\phi}{\partial t} : [0, T] \longrightarrow H^1,$$

such that

$$a(\phi(t), v) = \int_0^t < J(\phi(s)), v > ds + a(\phi_0, v), \quad \forall v \in \xi$$

Mathematical Formulation

and equivalently

$$(-\Delta + I)[\phi(t+\Delta t) - \phi(t)] = \int_{t}^{t+\Delta t} J(\phi(s))\, ds$$

The discrete scheme in time

$$\begin{cases} (-\Delta + I)[\Phi(t+\Delta t) - \Phi(t)] \simeq \Delta t J(\Phi_{\Delta t}(t)) \\ \\ \Phi(t) \text{ and } \Phi(t+\Delta t) \text{ are dpc} \\ \Phi_{\Delta t} \text{ is the approximation of } \phi \text{ at } \Delta t \end{cases}$$

Δ is discretized using a 5-point stencil

J will be carefully discretized in order to maintain in the FD scheme the quadratic quantities of physical importance (Mean square vorticity and Kinetic energy)

An explicit scheme will be used to evolve the system in time

Discretizing the Hasegawa Mima Equation

Discretizing the Hasegawa-Mima equation

$$\frac{\partial}{\partial t}\left(\nabla^2\phi - \phi\right) - \underbrace{\left[\phi, \nabla^2\phi\right] - \left[\phi, \ln\left(\frac{n_0}{\omega_{ci}}\right)\right]} = 0$$

The non-linear term is inside the Poisson bracket → **How to discretize the bracket ??**

We discretize the Poisson brackets using Arakawa’s Jacobian discretization

ILLUSTRATION

Vorticity (ζ) equation for two-dimensional incompressible flow:

(1) $$\frac{\partial \zeta}{\partial t} + \mathbf{v}\cdot\nabla\,\zeta = 0$$ where

$$\mathbf{v} = \mathbf{k}\times\nabla\,\psi,$$

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{v} \equiv \nabla^2\psi,$$

and ψ is the stream function, ∇ is the two-dimensional del operator, and $\mathbf{k}$ is unit vector normal to the plane of motion.

Equation (1) can be written as: $$\frac{\partial \zeta}{\partial t} = J(\zeta, \psi)$$

Integral Constraints on the Jacobian

There are the following integral constraints on the Jacobian:

$$\overline{J(\xi,\psi)} = 0, \qquad ①$$

$$\overline{\xi J(\xi,\psi)} = 0, \qquad ②$$

$$\overline{\psi J(\xi,\psi)} = 0, \qquad ③$$

① , ② and ③ imply respectively that:

The mean vorticity, $\overline{\zeta}$, the mean square vorticity, $\overline{\zeta^2}$ and the mean kinetic energy, $\frac{1}{2}\overline{(\nabla\psi)^2}$, are conserved with time.

Goal: Finite Difference (FD) for the Jacobian where we have

- → Conservation of Energy $K = \frac{1}{2}\overline{v^2}$
- → Conservation of Enstrophy $V = \frac{1}{2}\overline{\zeta^2}$

Conserving Energy & Enstrophy: Requirements for the maintenance of the last two constraints on the Jacobian

The FD analogue of the Jacobian at (*i*,*j*) may be written in general as:

$$\mathbb{J}_{i,j}(\zeta,\psi) = \sum_{i',j'} \sum_{i'',j''} c_{i,j;i',j';i'',j''}\, \zeta_{i+i',j+j'}\, \psi_{i+i'',j+j''}$$

It is convenient to define:

$$a_{i,j;\,i+i',j+j'} \equiv \sum_{i'',j''} c_{i,j;\,i',j';\,i'',j''}\, \psi_{i+i'',j+j''} \quad \text{and} \quad b_{i,j;\,i+i'',j+j''} \equiv \sum_{i',j'} c_{i,j;\,i',j';\,i'',j''}\, \zeta_{i+i',j+j'}$$

Then we have:

$$\mathbb{J}_{i,j}(\zeta,\psi) = \sum_{i',j'} a_{i,j;\,i+i',j+j'}\, \zeta_{i+i',j+j'} \quad \text{and} \quad \mathbb{J}_{i,j}(\zeta,\psi) = \sum_{i'',j''} b_{i,j;\,i+i'',j+j''}\, \psi_{i+i'',j+j''}.$$

To avoid false production of square vorticity and kinetic energy, we set the following requirements for the last two constraints to be maintained:

(2) is maintained if ⟹ $a_{i+i',j+j';\,i,j} = -a_{i,j;\,i+i',j+j'}$

(3) is maintained if ⟹ $b_{i+i'',j+j'';\,i,j} = -b_{i,j;\,i+i'',j+j''}$

A requirement for the maintenance of the first constraint on the Jacobian

Because a Jacobian vanishes if one of its arguments is a constant, we require that:

$$\sum_{i',j'} a_{i,j;\,i+i',j+j'} = 0 \quad \text{and} \quad \sum_{i'',j''} b_{i,j;\,i+i'',j+j''} = 0$$

The use of the condition on ② gives:

$$J_{i,j}(\xi,\psi) = \sum_{i',j'}{}^{*} \left(a_{i,j;\,i+i',j+j'}\, \xi_{i+i',j+j'} - a_{i-i',j-j';\,i,j}\, \xi_{i-i',j-j'} \right)$$

and

$$\sum_{i',j'}{}^{*} \left(a_{i,j;\,i+i',j+j'} - a_{i-i',j-j';\,i,j} \right) = 0.$$

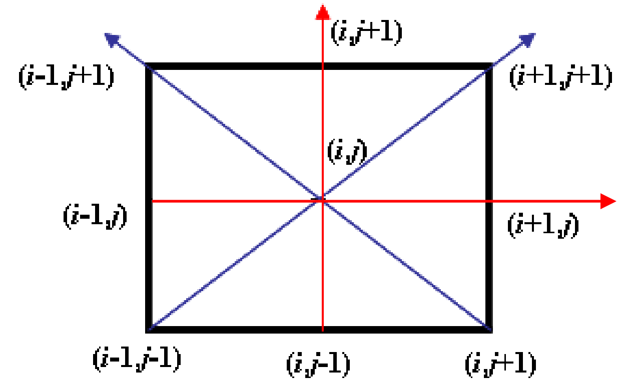
Combinations of the last two equations give a conservation form of the Jacobian that guarantees the maintenance of the first integral constraint.

Thus, ① is maintained if:

$$J_{i,j}(\xi,\psi) = \sum_{i',j'}{}^{*} \left[a_{i,j;\,i+i',j+j'} \left(\xi_{i+i',j+j'} - \xi_{i,j} \right) + a_{i-i',j-j';\,i,j} \left(\xi_{i,j} - \xi_{i-i',j-j'} \right) \right]$$

A Unique Scheme

Consider a **linear combination of the four basic finite difference analogues** for a square grid. We obtain by linear combinations of these four basic Jacobians a new Jacobian given by:



$$\mathbb{J}_{i,j}(\zeta,\psi) = \alpha\,\mathbb{J}^{++}_{i,j}(\zeta,\psi) + \beta\,\mathbb{J}^{+\times}_{i,j}(\zeta,\psi) + \gamma\,\mathbb{J}^{\times+}_{i,j}(\zeta,\psi) + \delta\,\mathbb{J}^{\times\times}_{i,j}(\zeta,\psi)$$

where $\alpha + \beta + \gamma + \delta = 1\,.$

The above expression of the Jacobian conserves mean square vorticity if

$$\alpha = \beta, \quad \delta = 0$$

and conserves energy if

$$\alpha = \gamma, \quad \delta = 0$$

By the choice of $\alpha,\ \beta,\ \gamma$ and δ one can obtain the forms of the Jacobian in the following table.

Properties of Typical Jacobians

$J(\xi,\psi) \Rightarrow$	J^{++}	$J^{+\times}$	$J^{\times+}$	$\frac{J^{++}+J^{+\times}}{2}$	$\frac{J^{+\times}+J^{\times+}}{2}$	$\frac{J^{\times+}+J^{++}}{2}$	$\frac{J^{++}+J^{+\times}+J^{\times+}}{3}$
$J(\xi,\psi) = -J(\psi,\xi)$	✓[α]				✓		✓
Square vorticity conserved			✓	✓			✓
Kinetic energy conserved		✓				✓	✓

[α] A check mark indicates that the property in the left-hand column is maintained.

The Final discrete Jacobian

- Preserves symmetry
- Conserves Energy
- Conserves Enstrophy

$$\alpha = \beta = \gamma = \tfrac{1}{3}, \quad \delta = 0.$$

$$\begin{aligned}
\mathbb{J}_{i,j}(\zeta,\psi) = -\frac{1}{12d^2}\big[&(\psi_{i,j-1} + \psi_{i+1,j-1} - \psi_{i,j+1} - \psi_{i+1,j+1}) \\
&(\zeta_{i+1,j} + \zeta_{i,j}) - (\psi_{i-1,j-1} + \psi_{i,j-1} - \psi_{i-1,j+1} - \psi_{i,j+1}) \\
&(\zeta_{i,j} + \zeta_{i-1,j}) + (\psi_{i+1,j} + \psi_{i+1,j+1} - \psi_{i-1,j} - \psi_{i-1,j+1}) \\
&(\zeta_{i,j+1} + \zeta_{i,j}) - (\psi_{i+1,j-1} + \psi_{i+1,j} - \psi_{i-1,j-1} - \psi_{i-1,j}) \\
&(\zeta_{i,j} + \zeta_{i,j-1}) + (\psi_{i+1,j} - \psi_{i,j+1})(\zeta_{i+1,j+1} + \zeta_{i,j}) \\
&- (\psi_{i,j-1} - \psi_{i-1,j})(\zeta_{i,j} + \zeta_{i-1,j-1}) + (\psi_{i,j+1} - \psi_{i-1,j}) \\
&(\zeta_{i-1,j+1} + \zeta_{i,j}) - (\psi_{i+1,j} - \psi_{i,j-1})(\zeta_{i,j} + \zeta_{i+1,j-1})\big].
\end{aligned}$$

Tests and Results

Nonlinear Solution for the HM Equation: The Modon Solution

The Hasegawa-Mima equation possesses a family of traveling wave solutions.

The simplest member of this family is a dipole vortex type solution, usually called a modon.

A stationary solution to the non-linear equation referred to as a modon is given by:

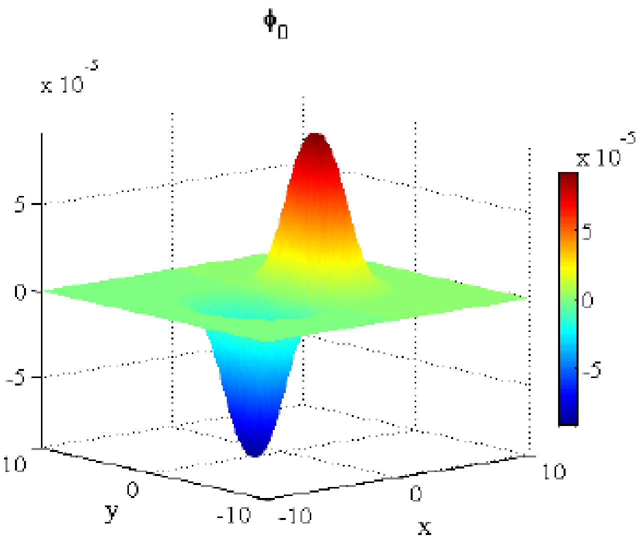
$$\Phi(r,\theta) = \begin{cases} \dfrac{acK_1(\beta r/a)}{K_1(\beta)}\, cos\theta, & \text{for } r > a \\ \\ ac[(1+\frac{\beta^2}{\gamma^2})\frac{r}{a} - \dfrac{\beta^2 J_1(\gamma r/a)}{\gamma^2 J_1(\gamma)}]\, cos\theta, & \text{for } r < a \end{cases}$$

where $r^2 = x^2 + (y - ct)^2$, $cos\theta = x/r$, $\beta = a(1+1/c)^{1/2}$, and a and c are parameters characterizing the size and speed of the modon, respectively.

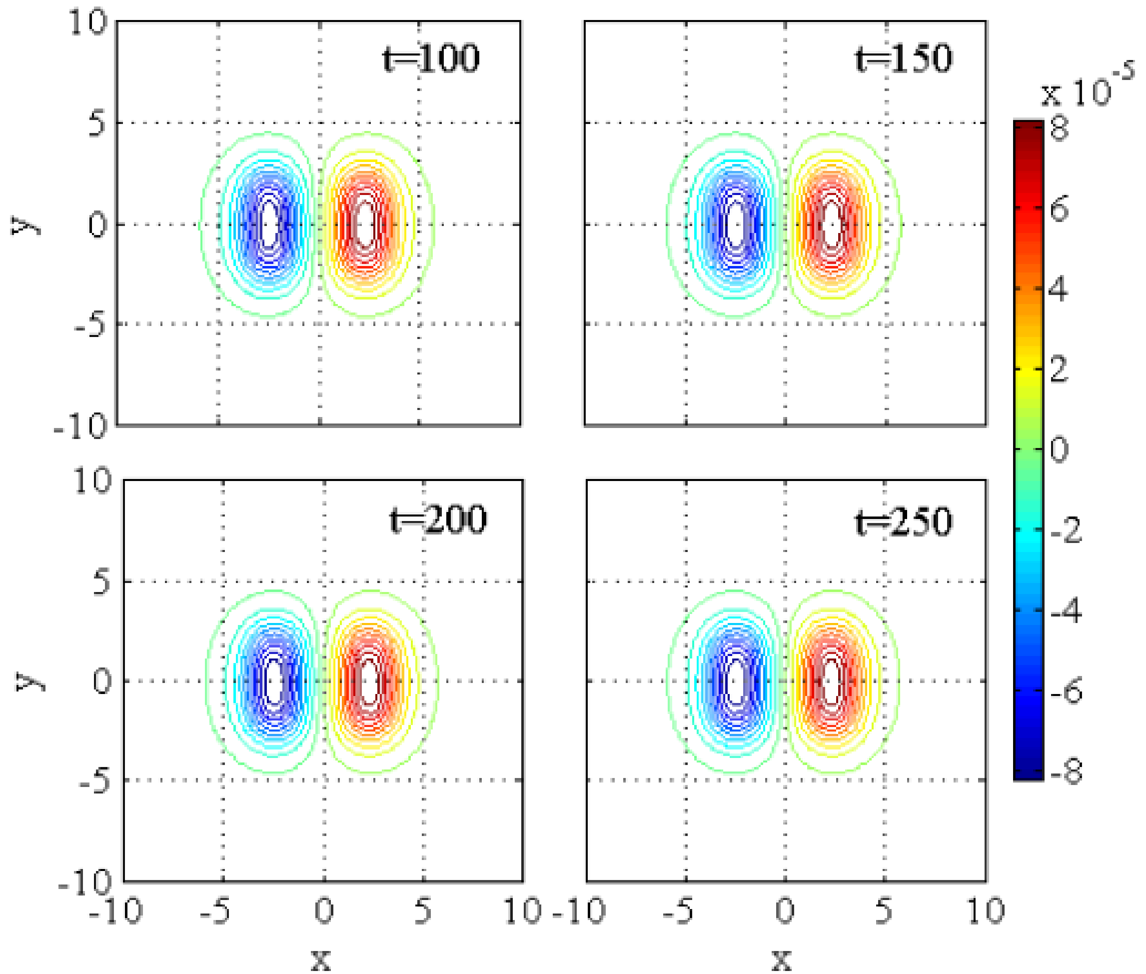
J_n and K_n are the n^{th} order Bessel function of the first kind and the second kind respectively. Here, γ is obtained through the continuity of the function at $r = a$,

$$K_2(\beta)/\beta K_1(\beta) = -J_2(\gamma)/\gamma J_1(\gamma)$$

Test for Stationarity

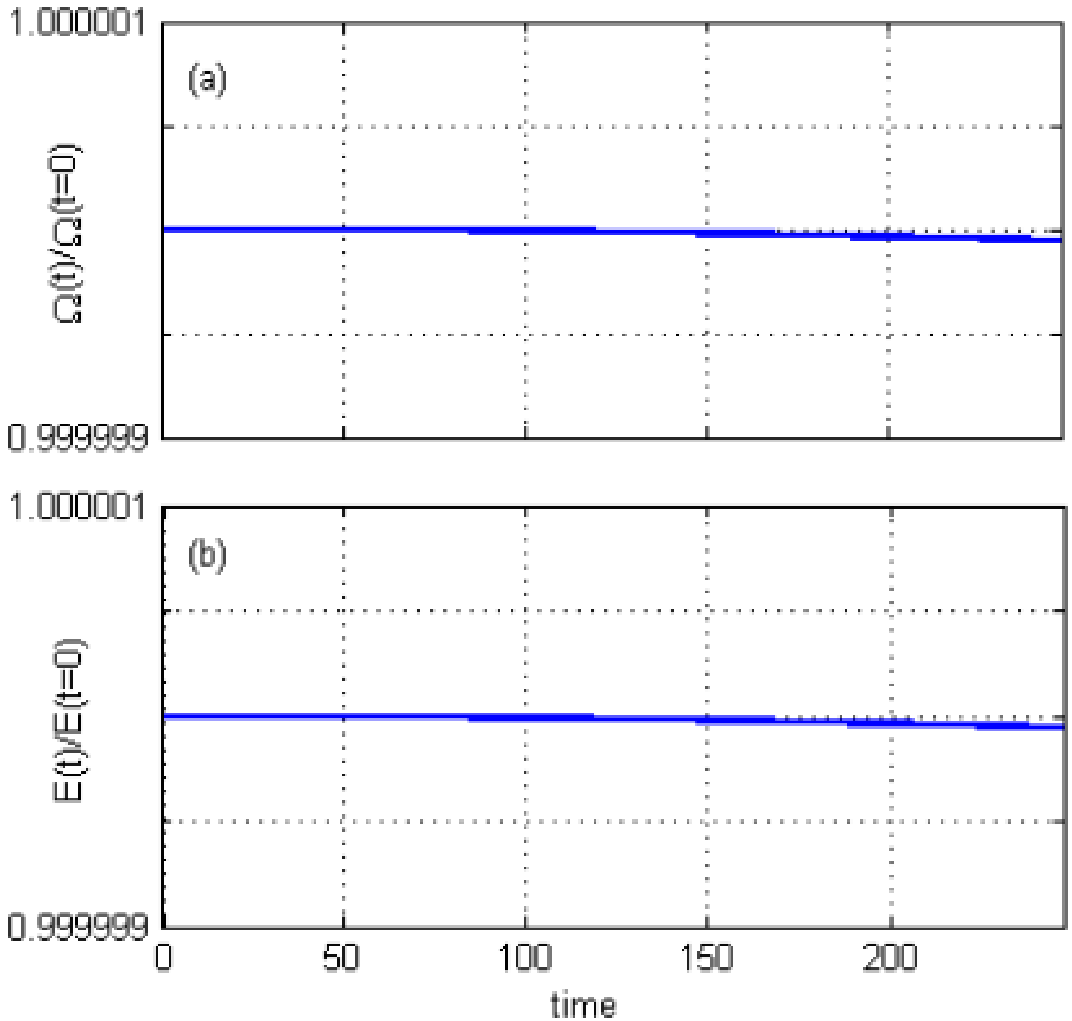


Initial potential profile for the modon dipole vortex solution with a = 2, c = 10 and $\gamma = 4.0914$.



Four contour plots of the electrostatic potential, the modon, taken at different times t, with a time step $\tau = 0.0496$ on a 128×128 grid.

Test for the conservation of generalized energy and enstrophy



For the Hasegawa-Mima equation, there are two conserved quantities.

The generalized energy

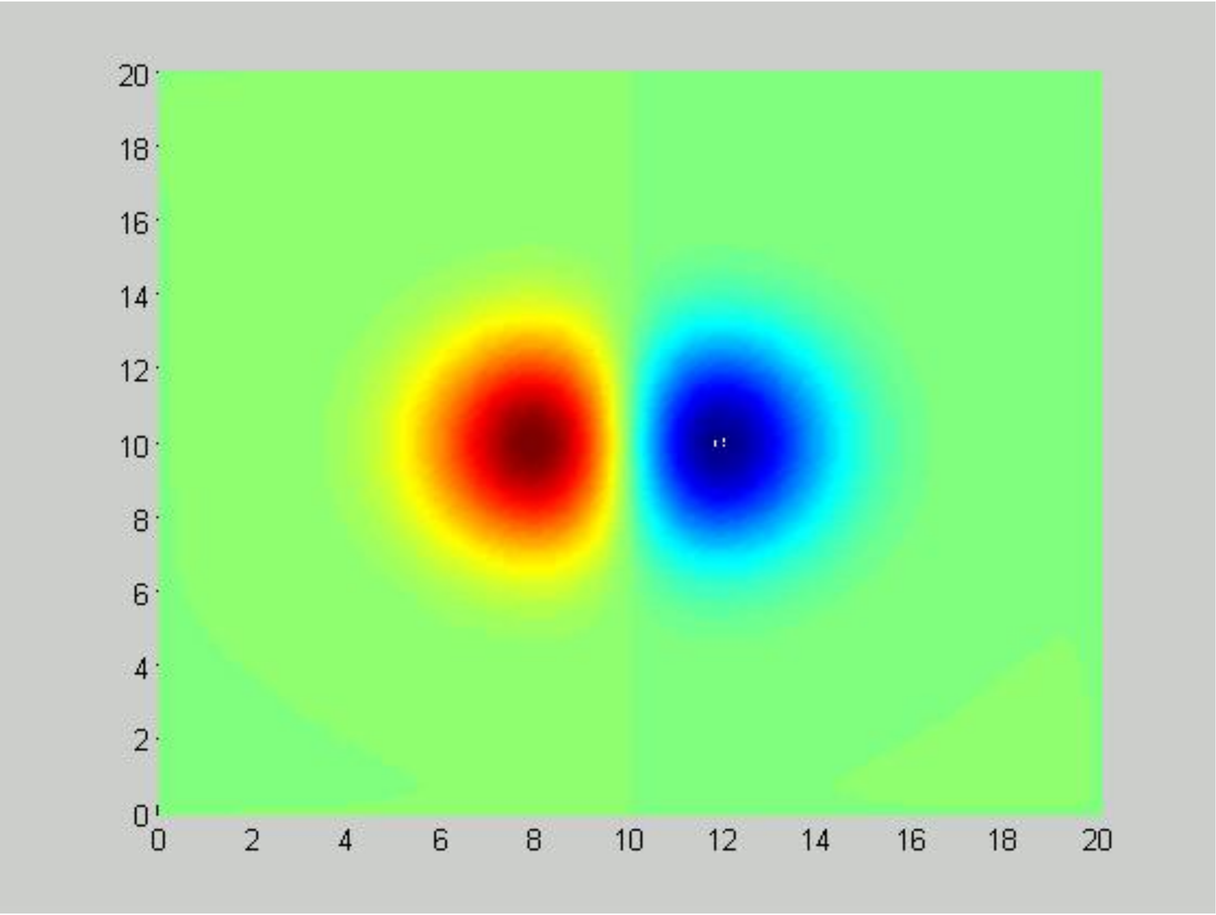
$$E[\phi] = \int [\phi^2 + (\nabla\phi)^2] dxdy,$$

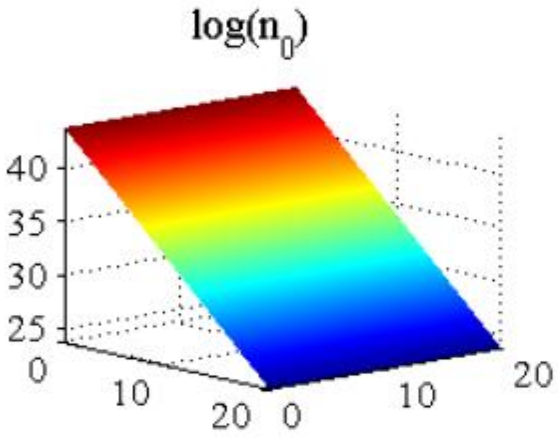
and the generalized enstrophy

$$\Omega[\phi] = \int [(\nabla\phi)^2 + (\nabla^2\phi)^2] dxdy.$$

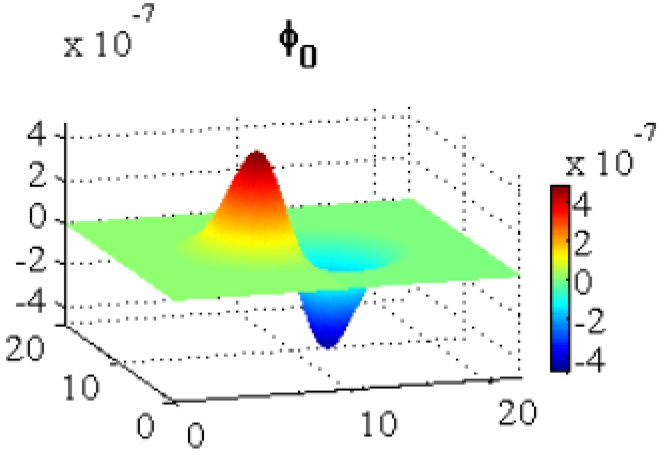
(a) Plot for the relative generalized enstrophy as a function of time
(b) Plot for the relative generalized energy as a function of time

Test for the Behavior of a Vortex Similar to a Modon and for Periodic Boundary Conditions





Plot of log(n0) where n0 is the initial density profile



Plot of the initial potential profile

Theory of Plane-Wave solutions

Let us recall the Hasegawa-Mima equation:

$$\partial_t(\Delta\phi - \phi) + \nabla_\perp\phi.\vec{\nabla}ln(n_0) + \nabla_\perp\phi.\vec{\nabla}(\Delta\phi) = 0.$$

A dispersion relation is necessary to know when the instabilities grow and at what rate.

We calculate the dispersion relation by linearizing the equation.

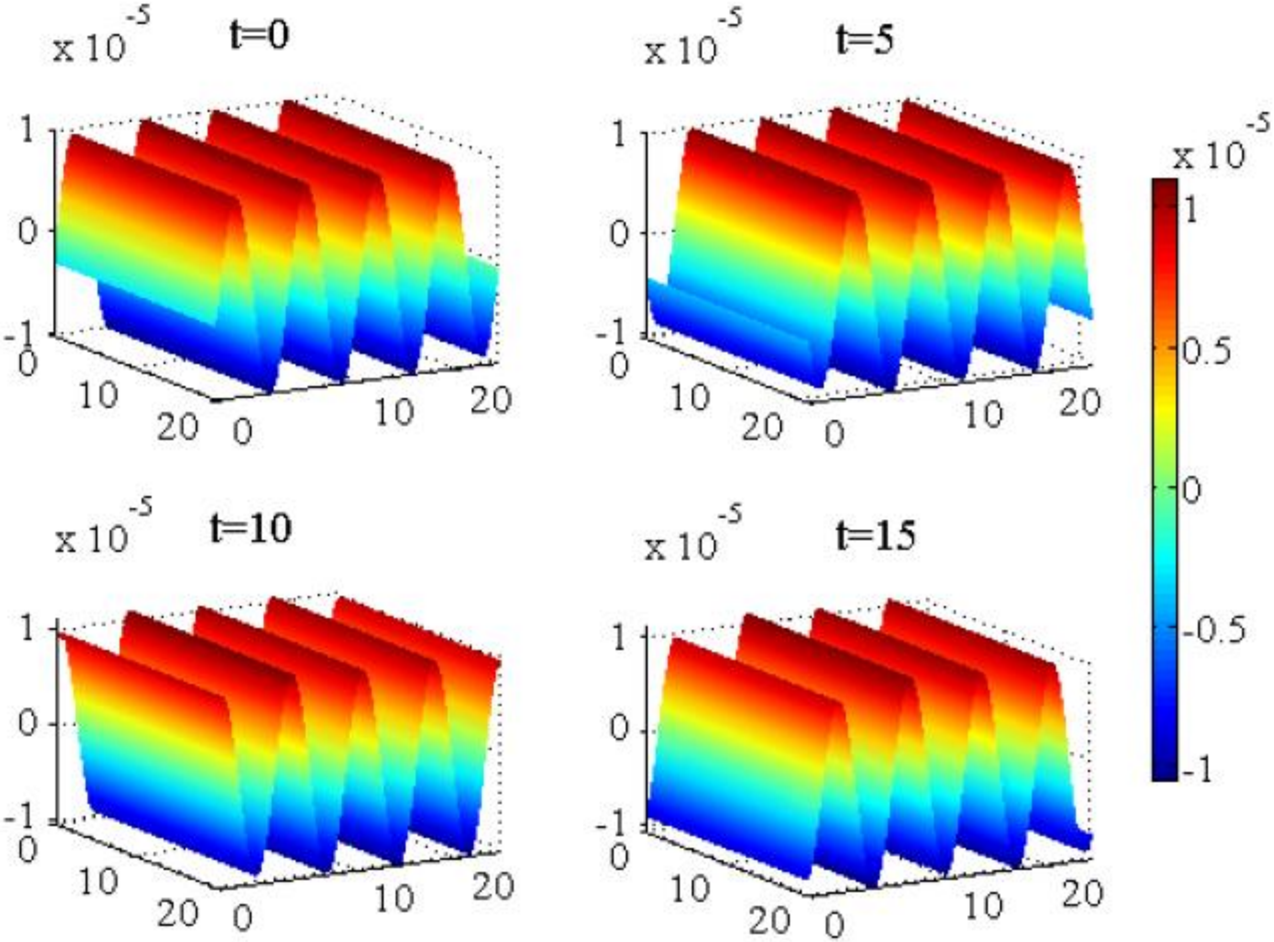
Consider a plane-wave solution of the type

$$\phi = \phi_k e^{ikx - i\omega t}$$

Then the dispersion relation is

$$\omega_* = -\frac{k\ .\ \vec{\nabla}lnn_0}{1+k^2} \in \mathbb{R} \implies \text{No instabilities}$$

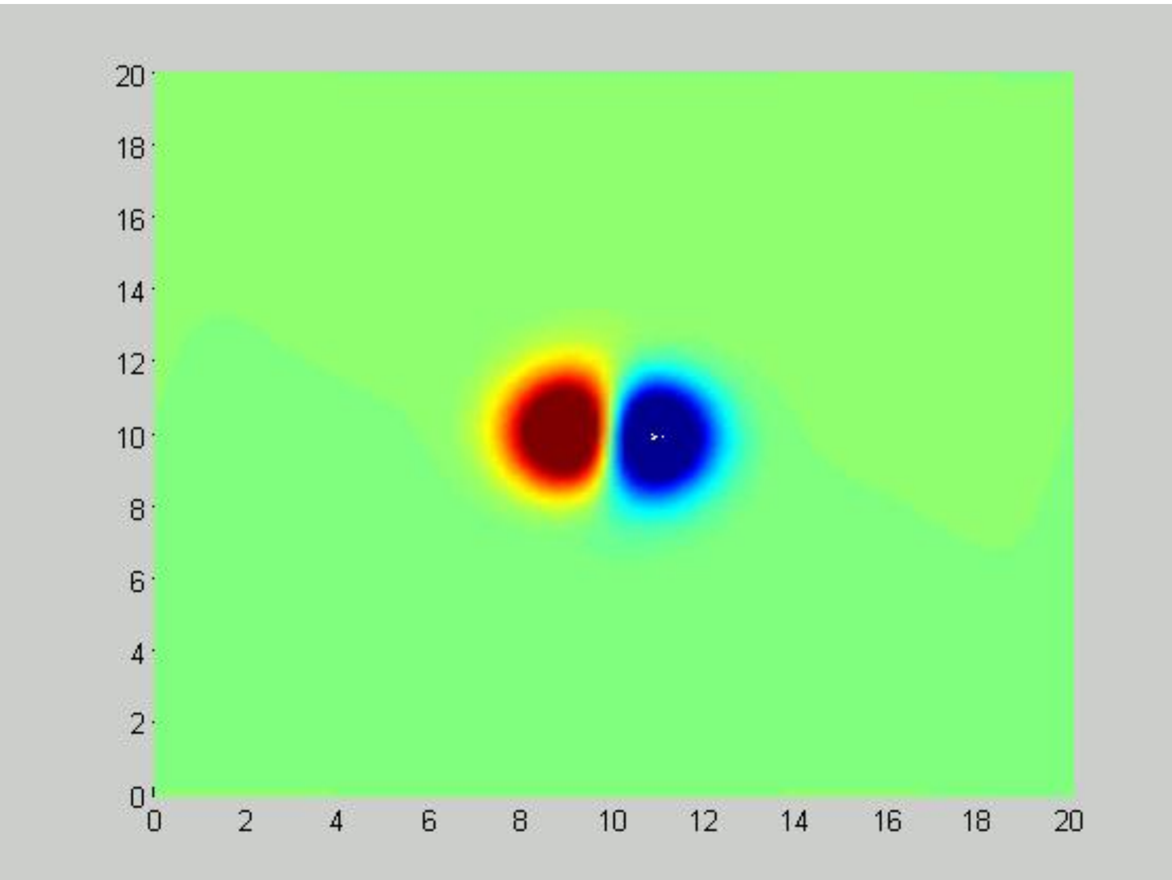
Test for a Plane-Wave solution: the Sine wave

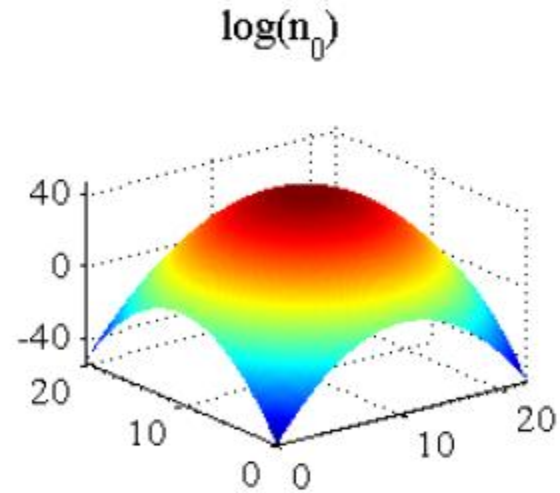


The time evolution of the potential when a plane-wave solution is considered

→ No instabilities are detected

Several Physics Problems that illustrate the richness of the dynamics depending on the initial conditions





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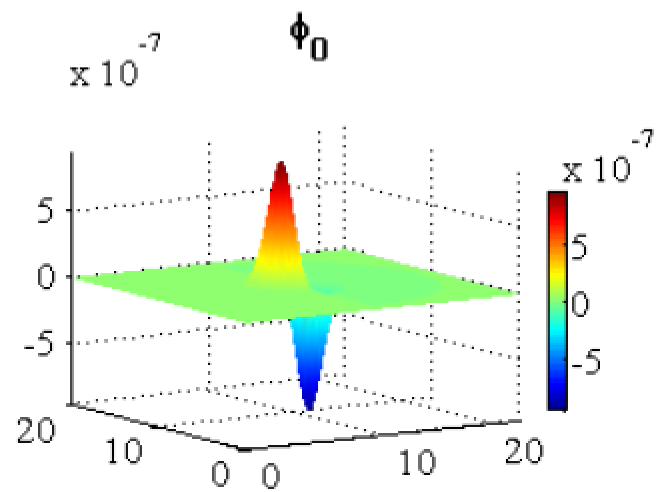
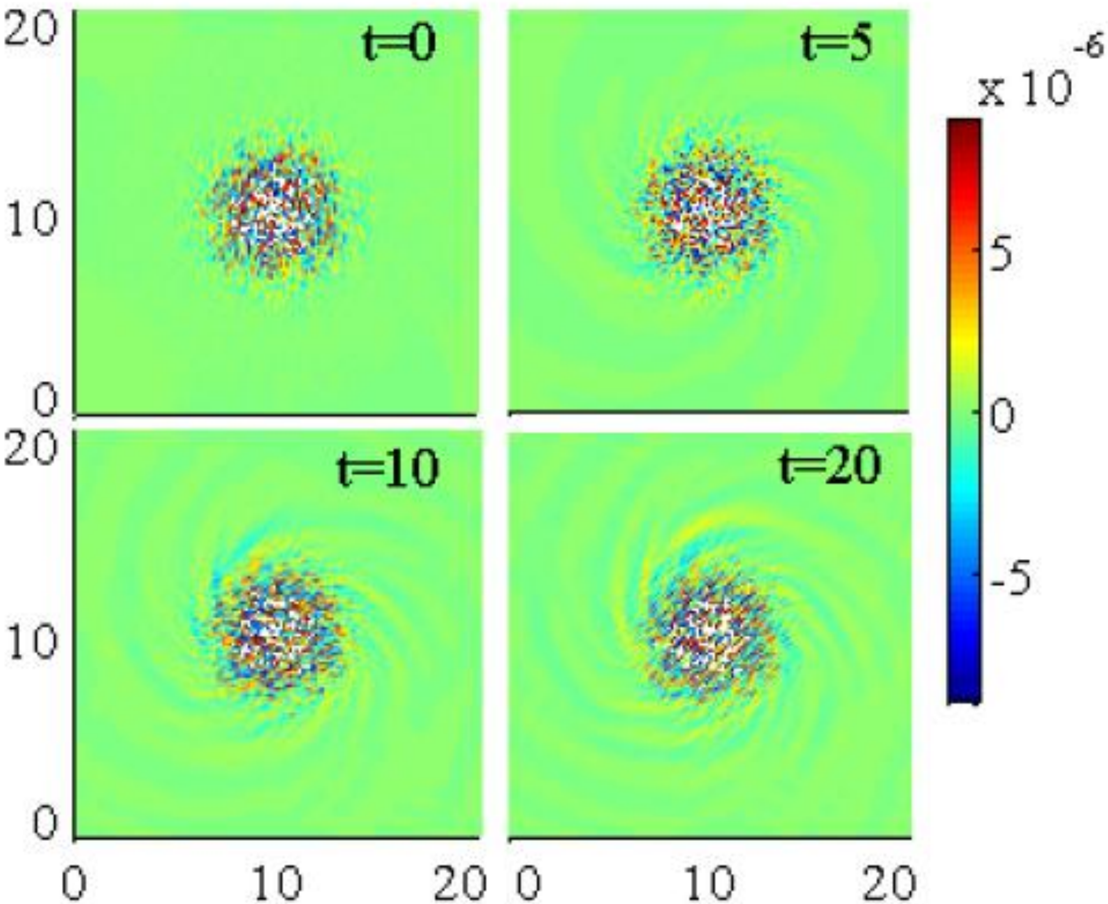
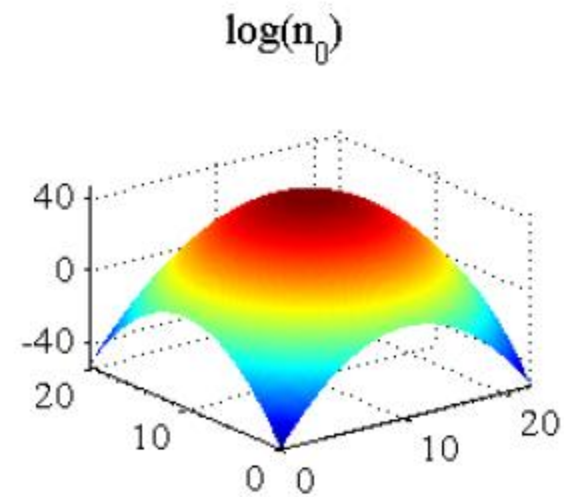


Plot of the initial potential profile

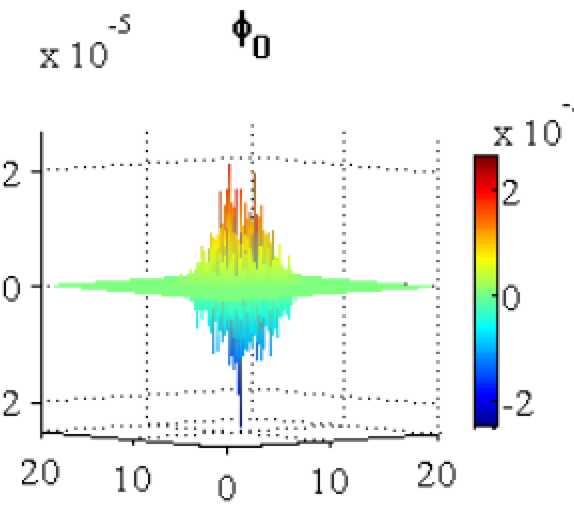
Several Physics Problems that illustrate the richness of the dynamics depending on the initial conditions



Time evolution of the electrostatic potential



Plot of log(n0) where n0 is the initial density profile



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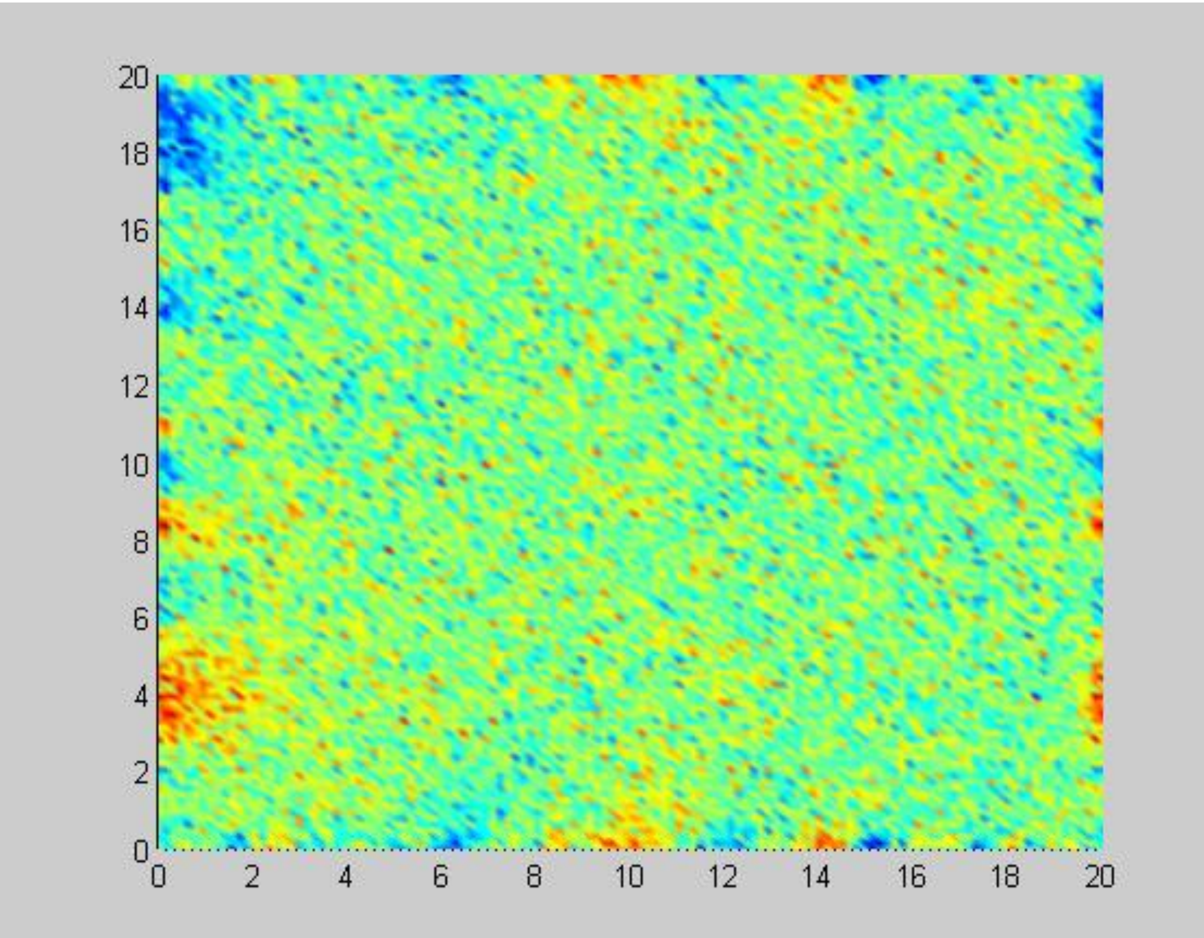

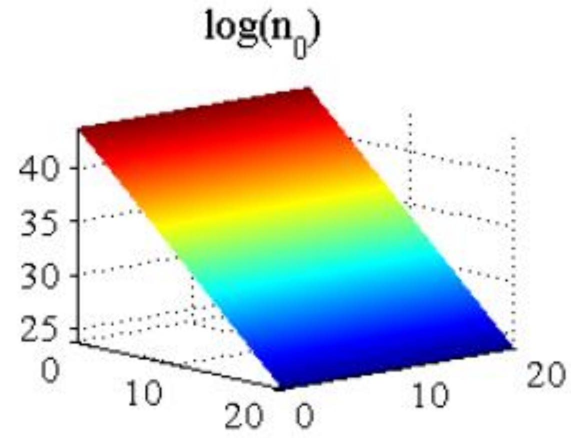

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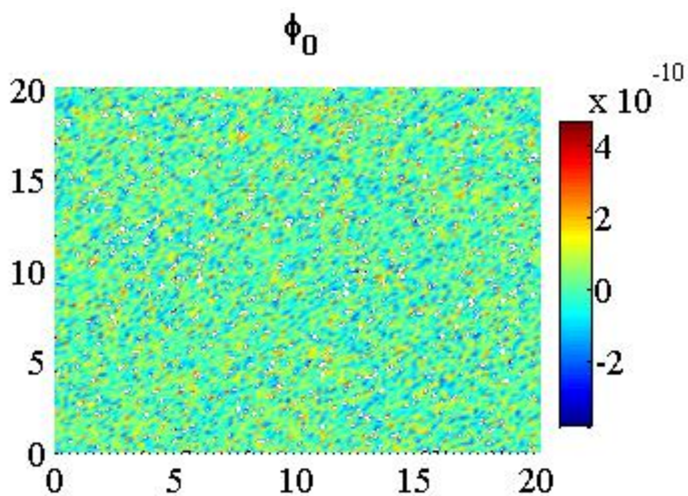

Plot of the initial potential profile

Summary and Future Work

- ✓ The Hasegawa-Mima model is a reduced form of the Navier-Stokes equations that simulates non-linear dynamics and turbulence in magnetically confined 2D plasmas

- ✓ The simulation is done using doubly periodic boundary conditions, Arakawa's method for the Jacobian discretization, and an explicit time discretization scheme.

- ✓ The correctness of the code is ensured by several tests

- ✓ Future work:
  - Study the dynamics of the results
  - Simulate another reduced plasma model that contains non-adiabatic electrons: The Hasegawa-Wakatani Model
  - Have the code parallelized to allow simulation of large devices