<u>Defn 1.</u> Let $u \in C^0(\mathbb{R}^m)$ (or $\mathcal{E}'(\mathbb{R}^m)$). The FBI transform of u is defined by

$$F(x,\xi) = \int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi - |\xi|\,|x-y|^2}\, u(y)\, dy,$$

$(x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m$.

<u>Thm 1.</u> (Inversion with the FBI). Let $u \in C^0(\mathbb{R}^m)$. Then

$$u(x) = \lim_{\epsilon \to 0^+} \frac{1}{(4\pi^3)^{m/2}} \iint_{\xi\ t} F(t,\xi)\, e^{i(x-t)\cdot\xi - \epsilon|\xi|^2}\, |\xi|^{m/2}\, dt\, d\xi.$$

<u>Pf:</u> From the Four. transf. of the Gaussian,

$$\int_{R^m} e^{i(x-y)\cdot\xi - \epsilon|\xi|^2}\, d\xi = \left(\frac{\pi}{\epsilon}\right)^{m/2} e^{\frac{-|x-y|^2}{4\epsilon}}.$$

Hence,

$$\frac{1}{(2\pi)^m} \iint e^{i(x-y)\cdot\xi - \epsilon|\xi|^2}\, u(y)\, d\xi\, dy = \frac{1}{2^m(\pi\epsilon)^{m/2}} \int e^{\frac{-|x-y|^2}{4\epsilon}}\, u(y)\, dy$$

$$= \frac{1}{\pi^{m/2}} \int e^{-t^2}\, u(x - 2\sqrt{\epsilon}\, t)\, dt$$

$$\longrightarrow u(x), \text{ uniformly}$$

on R^m since $u \in C^0(R^m)$.

Thus

$$u(x) = \lim_{\epsilon \downarrow 0} \frac{1}{(2\pi)^m} \iint e^{i(x-y)\cdot\xi - \epsilon|\xi|^2}\, u(y)\, d\xi\, dy$$

$$= \frac{1}{(4\pi^3)^{m/2}} \lim_{\epsilon \downarrow 0} \iiint e^{i(x-y)\cdot\xi - |\xi|\,|t-y|^2 - \epsilon|\xi|^2}\, |\xi|^{m/2}\, u(y)\, dt\, dy\, d\xi$$

$$= \frac{1}{(4\pi^3)^{m/2}} \lim_{\epsilon \downarrow 0} \iiint F(t,\xi)\, e^{i(x-t)\cdot\xi - \epsilon|\xi|^2}\, |\xi|^{m/2}\, dt\, d\xi$$

<u>Thm 2</u>. Let $u \in C^0(\mathbb{R}^m)$. TFAE:

(i) u is real analytic at $x_0 \in \mathbb{R}^m$.

(ii) $\exists$ a nbhd V of x_0 and constants $c_1, c_2 > 0$ $\ni$

$$|F(x,\xi)| \leq c_1 e^{-c_2|\xi|} \quad \forall (x,\xi) \in V \times \mathbb{R}^m.$$

<u>Pf</u>: (i) $\Rightarrow$ (ii). Spose u is real analytic at x_0. Let $0 \leq \varphi \leq 1$, $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi \equiv 1$ near x_0, supp $\varphi \subseteq \{x : u$ is real analytic at $x\}$.

The integrand in $F(x,\xi)$ has a holo. extension in a nbhd of $y = x_0$ in $\mathbb{C}^m$. Let u denote the extension of u near x_0. In the integration of $F(x,\xi)$, deform contour from $\mathbb{R}^m$ to $\Theta(\mathbb{R}^m)$ where $\Theta(y) = y - is\varphi(y)\frac{\xi}{|\xi|}$, s small enough so that u is defined on $\Theta(\mathbb{R}^m)$. Then

$$F(x,\xi) = \int_{\mathbb{R}^m} e^{Q(x,y,\xi)} u(\Theta(y)) \det \Theta'(y)\, dy, \text{ where}$$

$$Q(x,y,\xi) = i(x-\Theta(y))\cdot\xi - |\xi|(x-\Theta(y))^2. \text{ Note that}$$

$$\operatorname{Re} Q(x,y,\xi) = -s|\xi|\varphi(y)[1-s\varphi(y)] - |\xi||x-y|^2.$$

Let $\varphi(y) \equiv 1$ when $|y-x_0| \leq \delta$. Then with $s = \delta/4$,

$$|F(x,\xi)| \leq c\int_{|y-x_0|\leq\delta} e^{\operatorname{Re} Q(x,y,\xi)}\, dy + c\int_{\substack{|y-x_0|\geq\delta \\ y \in \operatorname{Supp}(u)}} e^{\operatorname{Re} Q(x,y,\xi)}\, d\xi$$

$$= I_1(x,\xi) + I_2(x,\xi).$$

$$I_1(x,\xi) \leq c\int_{|y-x_0|\leq\delta} e^{-s|\xi|\varphi(y)[1-s\varphi(y)]}\, dy = c\int_{|y-x_0|\leq\delta} e^{-\delta/4|\xi|[1-\delta/4]}\, dy \leq c' e^{-\delta/8|\xi|} \quad \forall\xi$$

choose $0 < \delta < 1$

$$I_2(x,\xi) \le c \int_{\substack{|y-x_0|\ge\delta, \\ y\in \mathrm{Supp}(u)}} e^{-|\xi||x-y|^2}\, dy. \quad \therefore \text{ when } |x-x_0| \le \delta/2$$

$$I_2(x,\xi) \le c'' e^{-\delta^2/4|\xi|}, \quad \forall \xi.$$

$\therefore$ for $|x-x_0| \le \delta/2$, $\xi \in \mathbb{R}^m$, $|F(x,\xi)| \le c_1 e^{-c_2|\xi|}$ for some $c_1, c_2 > 0$.

<u>(ii) $\Rightarrow$ (i)</u>. Assume WLOG $x_0 = 0$. Assume $|F(x,\xi)| \le c_1 e^{-c_2|\xi|}$ for x near 0, $\xi \in \mathbb{R}^m$. We'll use the inversion formula (Thm 1).

Write $\iint F(t,\xi)\, e^{i(x-t)\cdot\xi - \epsilon|\xi|^2} |\xi|^{m/2}\, dt\, d\xi = \sum_{j=1}^{4} I_j^\epsilon(x)$, where for some A_1, A_2, B to be chosen,

$I_1^\epsilon(x) =$ the integral over $\{(t,\xi) : |t| \le A_1, \xi \in \mathbb{R}^m\}$

$I_2^\epsilon(x) =$ " $\{(t,\xi) : A_1 \le |t| \le A_2, |\xi| \le B\}$

$I_3^\epsilon(x) =$ " $\{(t,\xi) : |t| \ge A_2, \xi \in \mathbb{R}^m\}$

$I_4^\epsilon(x) =$ " $\{(t,\xi) : A_1 \le |t| \le A_2, |\xi| \ge B\}$.

Will show: $\exists$ a nbhd of 0 in $\mathbb{C}^m$ to which the I_j^ϵ extend as holo fns and for each j, $I_j^\epsilon(z)$ converges uniformly on this nbhd as $\epsilon \to 0$.

Consider I_1^ϵ: Choose $A_1 > 0 \ni |F(x,\xi)| \le c_1 e^{-c_2|\xi|}$ for $(\bar{x},\xi)$, $|x| \le A_1$, $\xi \in R^m$.

In the integrand of I_1^ϵ, if we complexify x to $z = x+iy$, the integrand is bounded by a constant multiple of

$$|\xi|^{m/2} e^{(-c_2+|y|)|\xi|}$$ which has an integrable majorant for $|y| \le \frac{c_2}{2}$. Hence, as $\epsilon \to 0^+$, the entire fns $I_1^\epsilon(z)$ converge uniformly on a nbhd of 0 to a holo fn.

The fns I_2^ϵ easily extend as entire fns of z and converge uniformly on compact subsets to an entire fn as $\epsilon \to 0$.

Choose $A_2 > 0 \ni$ $\text{supp}(u) \subseteq \left\{ y : |y| \le \frac{A_2}{4} \right\}$.

$$F(t,\xi) = \int_{|y| \le \frac{A_2}{4}} e^{i(t-y)\cdot\xi - |\xi||t-y|^2} u(y)\, dy.$$

When $|t| \ge A_2$, $|t-y| \ge |t|-|y| = \frac{|t|}{2} + \frac{|t|}{2} - |y| \ge \frac{|t|}{2} + \frac{A_2}{4}$,

and so $|t-y|^2 \ge \frac{|t|^2}{4} + \frac{A_2^2}{16}$. Hence, when $|t| \ge A_2$

$$|F(t,\xi)| \le c\, e^{-|\xi|\left(\frac{|t|^2}{4} + \frac{A_2^2}{16}\right)}$$

This allows us to complexify as in I_1^ϵ to conclude that $I_3^\epsilon(z)$ converges uniformly to a holo fn in a nbhd of 0.

$$I_4^\epsilon(x) = \iiint_R e^{i(x-y)\cdot\xi - |\xi|\,|t-y|^2 - \epsilon|\xi|^2} |\xi|^{m/2} u(y)\,dy\,dt\,d\xi,$$

where $R = \{(y,t,\xi) : |\xi| \geq B,\ A_1 \leq |t| \leq A_2,\ y \in \mathrm{Supp}(u)\}$.

The fn $\xi \mapsto |\xi|$ $(\xi \neq 0)$, has a holo extension

$$\langle \zeta \rangle = \left(\sum_1^m \zeta_j^2\right)^{1/2} = e^{\frac{1}{2}\log\left(\sum_1^m \zeta_j^2\right)}$$

in the region $\zeta = \xi + i\eta$, $|\xi| > |\eta|$.

Change contour in the ξ-integration from $\mathbb{R}^m$ to the image under the map $\zeta(\xi) = \xi + is|\xi|(x-y)$ for $s > 0$, s small so that $|\mathrm{Im}\,\zeta(\xi)| < |\mathrm{Re}\,\zeta(\xi)|$ $(\xi \neq 0)$.

$$I_4^\epsilon(x) = \iiint_R e^{P(x,y,t,\xi,\epsilon)} \langle \zeta(\xi) \rangle^{m/2} u(y)\,dy\,dt\,d\xi,$$

where

$$P(x,y,t,\xi,\epsilon) = i(x-y)\cdot\xi - s|x-y|^2|\xi| - \langle \zeta(\xi) \rangle |t-y|^2 - \epsilon\,\zeta(\xi)^2.$$

For s small, $\mathrm{Re}\,\zeta(\xi)^2 \geq \frac{|\xi|^2}{2}$ and $\mathrm{Re}\,\zeta(\xi) \geq \frac{|\xi|}{2}$.

Hence

$$\left|e^{P(x,y,t,\xi,\epsilon)}\right| \leq e^{-s|x-y|^2|\xi| - \frac{|t-y|^2}{2}|\xi| - \epsilon/2\,|\xi|^2}.$$

In particular, when $x = 0$, since $|t| \geq A_1$, $\exists\, c > 0 \ni$

$$\left|e^{P(0,y,t,\xi,\epsilon)}\right| \leq e^{-c|\xi|} \quad \forall \xi.$$

This gives us enough freedom to complexify x to z and vary z near 0 to conclude that $I_4^\epsilon(z)$ converges to a holo fn on a nbhd of 0.

Thm 3 (Borel's lemma). Let $a_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{Z}^n$. Then $\exists f \in C^\infty(\mathbb{R}^n) \ni \frac{f^{(\alpha)}(0)}{\alpha!} = a_\alpha \; \forall \alpha$.

Pf: Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) \equiv 1$ on $B_{1/2}(0)$, $\operatorname{supp} \chi \subseteq B_1(0)$, $0 \le \chi \le 1$. Let $\{R_j\}_{j=1}^\infty$ be an increasing seq to be defined.

Set $$f(x) = \sum_\alpha a_\alpha \chi(R_{|\alpha|} x)\, x^\alpha$$

Continuity: $$|a_\alpha \chi(R_{|\alpha|}x) \cdot x^\alpha| \le |a_\alpha| \frac{1}{(R_{|\alpha|})^{|\alpha|}}$$

Choose $R_j > |a_\beta|$ for $|\beta| = j$

$$\le \frac{1}{(R_{|\alpha|})^{|\alpha|-1}} \le \frac{1}{R_1^{|\alpha|-1}}.$$ Choose $R_1 > 1$.

Smoothness. Let $\beta \in \mathbb{Z}^n$. Let $|\partial_x^\alpha \chi(x)| \le M_\alpha \; \forall \alpha$.

$$\left|a_\alpha \partial^\beta \{\chi(R_{|\alpha|}x)\, x^\alpha\}\right| = \left| a_\alpha \sum_{\gamma \le \beta,\, \gamma \le \alpha} \frac{\beta!}{\gamma!(\beta-\gamma)!} \partial_x^{\beta-\gamma} \chi(R_{|\alpha|}x)\, \partial_x^\gamma x^\alpha \right|$$

$$\le |a_\alpha| \sum_{\gamma \le \beta,\alpha} \frac{\beta!}{\gamma!(\beta-\gamma)!} R_{|\alpha|}^{|\beta-\gamma|} M_{\beta-\gamma} \frac{\alpha!}{(\alpha-\gamma)!} \frac{1}{R_{|\alpha|}^{|\alpha-\gamma|}}$$

$$= |a_\alpha| \sum_{\gamma \le \beta,\alpha} \frac{\beta!}{\gamma!(\beta-\gamma)!} \frac{R_{|\alpha|}^{|\beta|}}{R_{|\alpha|}^{|\alpha|}} M_{\beta-\gamma} \frac{\alpha!}{(\alpha-\gamma)!}$$

Choose $\{R_\gamma\}_\gamma$ $\quad R_\gamma \geq \left(M_{\gamma'}\right) \;\forall\; |\gamma'| = \gamma$

$$\leq \; |a_\alpha| \sum_{\gamma \leq \beta, \alpha} \frac{\beta!}{\gamma!\,(\beta-\gamma)!} \, \frac{R_{|\alpha|}^{|\beta|}}{\left(R_{|\alpha|}\right)^{|\alpha-\beta+\gamma|}} \, \frac{\alpha!}{(\alpha-\gamma)!}$$

$$\leq \sum_{\gamma \leq \beta, \alpha} \frac{\beta!}{\gamma!\,(\beta-\gamma)!} \, \frac{\alpha!}{(\alpha-\gamma)!} \, \frac{R_{|\alpha|}^{|\beta|+1}}{\left(R_{|\alpha|}\right)^{|\alpha-\beta+\gamma|}}$$

$$\leq \sum_{\gamma \leq \beta, \alpha} \frac{\beta!}{\gamma!\,(\beta-\gamma)!} \, \frac{\alpha!}{(\alpha-\gamma)!} \, \frac{R_{|\alpha|}^{2|\beta|+1}}{\left(R_{|\alpha|}\right)^{|\alpha|}}$$

$$\leq \beta! \sum_{\gamma \leq \alpha} \frac{\alpha!}{\gamma!\,(\alpha-\gamma)!} \, \frac{R_{|\alpha|}^{2|\beta|+1}}{\left(R_{|\alpha|}\right)^{|\alpha|}}$$

$$= \beta! \; 2^{|\alpha|} \, \frac{R_{|\alpha|}^{2|\beta|+1}}{\left(R_{|\alpha|}\right)^{|\alpha|}}.$$

(*) Exercise: Show that $\dfrac{f^{(\alpha)}(0)}{\alpha!} = a_\alpha \;\; \forall \alpha.$

<u>Theorem 4</u>. Let $f_0(x), f_1(x), \ldots$ C^∞ on $\mathbb{R}^n$. Then $\exists\, F(x,y)$ C^∞ on $\mathbb{R}^n \times \mathbb{R}$ $\ni$ $\frac{\partial_y^k F(x,0)}{k!} = f_k(x)$ $\forall k$.

$\left(F(x,0) = f_0(x) \right)$. $K \subseteq \mathbb{R}^n$ cpt.

<u>Pf</u>: $$F(x,y) = \sum_{j=0}^{\infty} f_j(x)\, \chi(R_j y)\, y^j, \quad \chi \text{ as before,}$$

the R_j to be determined, $\{R_j\}$ increasing, $R_1 > 1$.

<u>Continuity</u>: Easy.

Fix β and n.

$$\left| \partial_x^\beta \partial_y^n \left[f_j(x)\, \chi(R_j y)\, y^j \right] \right|$$

$$= \left| \partial_x^\beta f_j(x) \right| \left| \sum_{\substack{k=0,\\ k \le j}}^{n} \binom{n}{k} \partial_y^{n-k} \chi(R_j y)\, \partial_y^k (y^j) \right|$$

$$\le M_j^\beta \sum_{k \le n, j} \frac{n!}{k!(n-k)!} R_j^{n-k} L_{n-k} \frac{j!}{(j-k)!} \cdot \frac{1}{R_j^{j-k}}$$

$$= M_j^\beta \sum_{k \le n, j} \frac{n!}{k!(n-k)!} R_j^{n-j} L_{n-k} \frac{j!}{(j-k)!}.$$

May assume $|\beta| + n \le j$. Choose $R_j \ni$

$M_j^\beta \le R_j$ for $|\beta| \le j$ and $L_i \le R_i$ $\forall i$, R_i incr, R_1

Then $$\le \sum_{k \le n, j} \frac{n!}{k!(n-k)!} R_j^{n-j+2} \frac{j!}{(j-k)!}$$

$$\le n! \sum_{k=0}^{j} R_j^{\,n-j+2} \frac{j!}{k!(j-k)!}$$

$$\le n!\, 2^j R_j^{\,n-j+2} \quad \therefore \text{ Done.}$$

~~Theorem 4.~~ & (*) Exercise: Show that $\dfrac{\partial_y^k F(x,0)}{k!} = f_k(x)$.

(*) Exercise: Let $\{ f_\alpha(x) : \alpha \in \mathbb{Z}^n \}$ be C^∞ on $\mathbb{R}^n$.

Show $\exists\, F(x,y) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ C^∞ $\ni$

$$\frac{\partial_y^\beta F(x,0)}{\beta!} = f_\beta(x) \quad \forall \beta \in \mathbb{Z}^n.$$

~~Theorem.~~ Notation: If $F : \mathbb{R}^n_x \times \mathbb{R}^n_y \longrightarrow \mathbb{R}$ is C^1,

$$\frac{\partial F}{\partial \bar{z}_j}(x,y) = \frac{1}{2}\left(\frac{\partial F}{\partial x_j} + \sqrt{-1}\, \frac{\partial F}{\partial y_j} \right), \quad 1 \le j \le n$$

F holo on $\mathbb{C}^n_z \sim \mathbb{R}^n_x \times \mathbb{R}^n_y$ if $\dfrac{\partial F}{\partial \bar{z}_j} = 0 \ \ \forall 1 \le j \le n$.

Lemma 5. Let $f(x)$ be C^∞ on $\mathbb{R}^n$. Then $\exists\, F(x,y)$ C^∞ on $\mathbb{R}^n_x \times \mathbb{R}^n_y \ni$ (1) $F(x,0) = f(x)$; (2)

$$\frac{\partial F}{\partial \bar{z}_j}(x,y) = O(|y|^N) \ \ \forall N.$$

Such an F is called an almost analytic extension of F.

Proof: Want $F(x,0)=f(x)$ and $\dfrac{\partial F}{\partial \bar{z}_j}(x,y)=O(|y|^N)$

$$\Longleftrightarrow \quad \partial_y^\beta \left\{ \frac{\partial F}{\partial \bar{z}_j}(x,y) \right\} \Big|_{y=0} = 0 \quad \forall \beta \quad (\text{Exercise}).$$

$$\frac{\partial F}{\partial \bar{z}_j}(x,y) = \frac{1}{2}\left(\frac{\partial F}{\partial x_j}(x,y) + \sqrt{-1}\,\frac{\partial F}{\partial y_j}(x,y) \right)\Big|_{y=0} = 0 \Rightarrow$$

$$\frac{\partial F}{\partial y_j}(x,0) = \sqrt{-1}\,\frac{\partial f}{\partial x_j}(x,0) \quad \forall j.$$

Induction: suppose $\partial_y^\beta F(x,0)$ is determined ~~sat~~ in terms of $f(x)$ and the derivatives of f. Then

$$\partial_y^\beta \left(\frac{\partial F}{\partial \bar{z}_j} \right)\Big|_{y=0} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\partial_y^\beta F + \sqrt{-1}\,\partial_y^{\beta+e_j} F \right)\Big|_{y=0} = 0$$

$$\Rightarrow \quad \partial_y^{\beta+e_j} F(x,0) = \sqrt{-1}\,\frac{\partial}{\partial x_j}\,\partial_y^\beta F(x,0).$$

Thus $\partial_y^\beta F(x,0) = f_\beta(x)$ are determined, $f_0(x)=f(x)$. Then use the exercise.

Check it works.

Theorem 6. Let f be cont. on $\mathbb{R}^n$. Then f is C^∞.

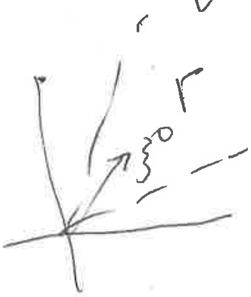
$\Longleftrightarrow$ $\exists\, F(x,y)$ C^1 on $\mathbb{R}^n \times \mathbb{R}^n$ that is an almost analytic extension of $f(x)$.

Pf of Thm 6 $\Rightarrow$ done with F C^∞.

$\Leftarrow$. Let $x_0 \in \mathbb{R}^m$. Let $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi \equiv 1$ near x_0, Supp $\varphi \subseteq B_R(x_0)$. Let $\Phi(x,y)$ alm anal ext of $\varphi(x)$.

~~Let $D = \{($~~ Fix $\xi^0 \in \mathbb{R}^m$, $|\xi^0| = 1$.

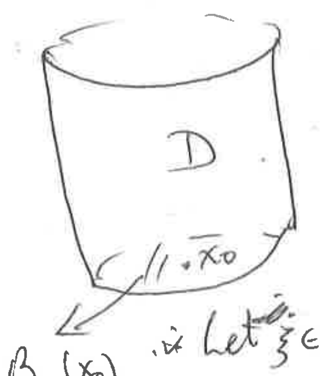
Let $y_0 \in \mathbb{R}^m$ $\ni$ $y_0 \cdot \xi^0 < -c < 0$. Then $\exists\ \delta > 0$ $\ni$ $|\eta| = 1$ and $|\eta - \xi^0| < \delta \Rightarrow y_0 \cdot \eta < -c/2$.



Let $\Gamma = \left\{\xi : \left|\frac{\xi}{|\xi|} - \xi^0\right| < \delta\right\}$ ~~Fix $\xi \in$~~

Then if $\xi \in \Gamma$, $y_0 \cdot \xi = \left(y_0 \cdot \xi/|\xi|\right)|\xi| < -\frac{c}{2}|\xi|$.

Let $D = \{(x, ty_0) : |x - x_0| < R,\ 0 \le t \le 1\}$



$$\int_{\partial D} g(z)\, dz_1 \wedge \cdots \wedge dz_m = \sum_j \iint_D \frac{\partial g}{\partial \bar z_j}\, d\bar z_j \wedge d$$

Let $\xi \in \Gamma$. $\displaystyle\int_{\partial D} e^{-iz\cdot\xi}\, \Phi(z)\, F(z)\, dz_1 \wedge \cdots \wedge dz_m$ $\quad z = (x,y)$

$$= \iint_D e^{-iz\cdot\xi}\left[\frac{\partial \Phi}{\partial \bar z_j} F + \Phi \frac{\partial F}{\partial \bar z_j}\right] d\bar z_j \wedge dz$$

$z = x + i t y_0$

Wave front sets

1. C^∞ WF set: Let $u \in C(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$. Let $\xi^0 \in \mathbb{R}^n \setminus 0$. We say $(x_0, \xi^0) \notin WFu$ if for some $\varphi \in C_0^\infty(\Omega)$, ~~$\varphi(x_0)$~~ $\varphi \equiv 1$ near x_0, $\exists$ a conic nbhd Γ of ξ^0 $\ni$ $|\widehat{\varphi u}(\xi)| \leq \frac{C_k}{|\xi|^k}$ $\forall \xi \in \Gamma$.

Remark: u is C^∞ at $x_0 \iff (x_0, \xi^0) \notin WFu$ $\forall \xi \in \mathbb{R}^n \setminus 0$
$\iff (x_0, \xi) \notin WFu$ $\forall \xi \in \mathbb{R}^n \setminus 0, |\xi| = 1$

2. C^ω WF set: u as above. We say $(x_0, \xi^0) \notin WF_a u$ if for some φ as above, $\exists$ Γ of ξ^0 and V of x_0 $\ni$

$$|F(x,\xi)| \leq C_1 e^{-C_2|\xi|} \quad \forall x \in V, \xi \in \Gamma.$$

~~$\iff \xi \in \Gamma, |\xi$~~

Remark. u is C^ω at $x_0 \iff (x_0, \xi) \notin WF_a u$ $\forall \xi \in \mathbb{R}^n$
$\iff (x_0, \xi) \notin WF_a u$ $\forall \xi \in \mathbb{R}^n \setminus 0, |\xi| = 1$

2': $(x_0, \xi^0) \notin WF_a u \iff \exists$ a nbhd V of x_0, and disjoint open ~~cones~~ acute cones $\Gamma^1, \dots, \Gamma^N \subseteq \mathbb{R}^n \setminus 0$ $\ni$ $\xi^0 \cdot \Gamma^j < 0$ $\forall j$, and holo fns F_j defined on truncated wedges $V + \sqrt{-1}\,\Gamma^j_\delta = \{x + \sqrt{-1}\, y : x \in V, y \in \Gamma^j, |y| < \delta\}$ $\ni$

$$u(x) = \sum_{j=1}^{N} F_j(x) \text{ on } V.$$

Theorem 7. Let u be cont near $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m \setminus 0$. Then

$(x_0, \xi^0) \notin WF(u) \iff \exists$ a nbhd V of x_0, cones Γ^j $1 \le j \le N$, $\xi^0 \cdot \Gamma^j < 0$ $\forall j$,
almost analytic fns $u_j(x,y)$ on $V + i\Gamma^j$ $\ni$

$$u = \sum_{j=1}^{N} u_j \text{ on } V.$$

Pf: $\Rightarrow$. Spose $(x_0, \xi^0) \notin WF(u)$. Let $\mathcal{C}_j$, $1 \le j \le N$ be disjoint acute cones, $\xi^0 \in \mathcal{C}_1$, $\overline{\mathcal{C}}_j \cap \overline{\mathcal{C}}_k$ of measure 0, $j \ne k$

For $j \ge 2$, Since $\xi^0 \notin \mathcal{C}_j$, $\exists$ a cone $\Gamma^j \ni \xi^0 \cdot \Gamma^j < 0$ and $\mathcal{C}_j \cdot \Gamma^j > 0$. May assume, $v \cdot \xi \ge c_j |v||\xi|$, $v \in \mathcal{C}_j$, $\xi \in \Gamma^j$, $c_j > 0$. For $j \ge 2$, consider

$$u_j(x+iy) = \int_{\mathcal{C}_j} e^{i(x+iy)\cdot\xi}\, \hat{u}(\xi)\, d\xi, \quad y \in \Gamma^j.$$

$$\cancel{\text{Since}}\ \operatorname{Re}(i(x+iy)\cdot\xi) = -y\cdot\xi \le -c_j|y||\xi|.$$

$\therefore$ $u_j(x+iy)$ is holo on $\mathbb{R}^n + i\Gamma^j$, $j \ge 2$.

~~$u_1(x)$~~ Assume that on $\mathcal{C}_1$, $|\hat{u}(\xi)| \le \dfrac{c_k}{|\xi|^k}$ $\forall k$.

Then $u_1(x) = \displaystyle\int_{\mathcal{C}_1} e^{ix\cdot\xi}\, \hat{u}(\xi)\, d\xi$ is C^∞,

$\therefore$ it has an almost analytic ext. on $\mathbb{R}^n + i\mathbb{R}^n$.

$\Leftarrow$ Suppose $u = \sum_{j=1}^{N} u_j$, ~~u_j~~ $u_j(x,y)$ almost analytic on $V + i\Gamma^j$, $\xi^0 \cdot \Gamma^j < 0$.

Use the pf of $\Leftarrow$ in Thm 6 by using $y_0 \in \Gamma^j$.

Thm 8. Let u be cont near $x_0 \in \mathbb{R}^m$, $\xi^0 \in \mathbb{R}^m \setminus 0$. Then $(x_0, \xi^0) \notin WF_a(u) \Leftrightarrow \exists$ a nbhd V of x_0, cones Γ^j, $1 \le j \le N$, $\xi^0 \cdot \Gamma^j < 0$ $\forall j$, and holo fns $u_j(x+iy)$ on $V + i\Gamma^j_\delta$ $\ni$

$$u = \sum_{j=1}^{N} u_j \text{ on } V.$$

Pf: Same as the one of Thm 8, using the FBI transform.

Corollary 9: $WF u \subseteq WF_a u$.

Coro 10. The FBI transform can be used to characterize $WF(u)$.

Pf: Use Thm ~~8~~ 7.

Example: The case $n=1$.

$1'$: $(x_0,\xi^0) \notin WFu \Leftrightarrow \cdots\cdots$ F_j C^1 on $V+\sqrt{-1}\,\Gamma_\delta^j$, almost analytic, and $u(x) = \sum_{j=1}^{N} F_j(x)$

Example: The case $n=1$.

(*) This is a refined study of singularities.

Motivation:

$$P(x,D) = \sum_{|\alpha|\le L} a_\alpha(x) D^\alpha \text{ in } \Omega \underset{\text{open}}{\subseteq} \mathbb{R}^n.$$

~~$Pu = f$~~. $p_L(x,\xi) = \sum_{|\alpha|=L} a_\alpha(x)\xi^\alpha$ the princ. symbol

~~(1) If f P is C^∞, and f is C^∞~~,

The characteristic set of P,

$$\operatorname{Char} P = \{(x,\xi)\in \Omega\times \mathbb{R}^n\setminus 0 : p_m(x,\xi)=0\}.$$

P is called elliptic if $\operatorname{Char} P = 0$

eg: Δ, $\frac{\partial}{\partial \bar z_j}$.

Thm A: P C^∞. If $Pu=f$, then
$$WFu \subseteq \operatorname{Char} P \cup WF(f).$$

* Eg: The wave eqn.

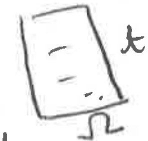
Coro 1: $Pu=f$, f $C^\infty \Rightarrow WFu \subseteq \operatorname{Char} P$.

Coro 2: P elliptic, f $C^\infty \Rightarrow WFu = \phi \Rightarrow u$ is C^∞.

Thm B: C^ω case.

Thm 3: G^s. Gevrey.

<u>Lemma</u>. Let $\Omega \subseteq \mathbb{R}^m$ be open and $U = \Omega_x \times (0, \delta)_t$.



Let $L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x,t) \frac{\partial}{\partial x_j}$ be a C^1 vector field on U. Assume $f \in C^1(\overline{U})$ and

$$Lf(x,t) = O(t^k)\ \forall k \quad (f \text{ is an approx soln}).$$

S'pose $\exists$ C^1 fns $\varphi_1(x,t), \ldots, \varphi_m(x,t)$ on $\overline{U}$ $\ni$

$$Z_j(x,t) = x_j + t\varphi_j(x,t) \text{ satisfy}$$

$$LZ_j(x,t) = O(t^k)\ \forall k.$$

Let $x_0 \in \mathbb{R}^n$, $\xi^0 \in \mathbb{R}^n \setminus 0$. Assume that

$$\langle \operatorname{Im} a(x_0, \xi^0), \xi^0 \rangle > 0.$$

$$\langle \operatorname{Im} a(x_0,\xi^0), \xi^0\rangle > 0.$$

Then $(x_0, \xi^0) \notin WF(f(x,0))$.

<u>Remark</u>: If L is C^∞, the φ_j exist.

<u>Pf</u>: WLOG, assume $x_0 = 0$.

Let $M_j = \sum_{\ell=1}^{m} b_{j\ell}(x,t) \frac{\partial}{\partial x_\ell}$, $1 \le j \le m$ $\ni$

$$M_j Z_k = \delta_j^k = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}.$$

$$M_j Z_k = \sum_{\ell=1}^{m} b_{j\ell} \frac{\partial Z_k}{\partial x_\ell} = \delta_j^k \iff$$

$$BZ_x = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix} \begin{pmatrix} \frac{\partial Z_1}{\partial x_1} & \cdots & \frac{\partial Z_m}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial Z_1}{\partial x_m} & \cdots & \frac{\partial Z_m}{\partial x_m} \end{pmatrix} = Id.$$

[illegible]. 17

1. $\{M_1, \ldots, M_m\}$ are lin indep. at each pt.

<u>Pf</u>: $\sum_{l=1}^{m} A_l M_l = 0 \Rightarrow A_k = \left(\sum_{l=1}^{m} A_l M_l\right)(Z_k)$
$= 0.$

2. $[M_j, M_k] = 0$ <u>Pf</u>: Since $\{M_l\}$ is a basis,

$$[M_j, M_k] = \sum_{l=1}^{m} B_l M_l .$$

$$\Rightarrow 0 = [M_j, M_k](Z_i) = \left(\sum_l B_l M_l\right)(Z_i) = B_i$$

$$\therefore [M_j, M_k] = 0 .$$

3. $[M_j, L] = \sum_{s=1}^{m} c_{js} M_s$ ~~where~~ with

$c_{js} = O(t^k)\ \forall k.$

<u>Pf</u>: Let $[M_j, L] = \cancel{A_0 L + \sum_{j=1}^{m} A_j M_j}\ c_0 L + \sum_{s=1}^{m} c_{js} M_s$

Apply to the fn t: ~~$A_0 = 0$ or~~ $c_0 = 0$.

$\therefore$ ~~Either~~ $[M_j, L] = \sum_{s=1}^{m} c_{js} M_s .$

Apply to Z_k: $M_j(L Z_k) = c_{jk}$

$\Rightarrow c_{jk} = O(t^l)\ \forall l.$

If $g(x,t)$ is a C^1 fn,

$$dg = \sum_{k=1}^{m} M_k(g)\, dZ_k + \Big(Lg - \sum_{k=1}^{m} M_k(g) LZ_k \Big) dt \quad (*)$$

This is verified by evaluating each side at the basis of vector fields $\{L, M_1, \ldots, M_m\}$.

Using $(*)$, we get

$$(**)\quad d\left(g\, dZ_1 \wedge \cdots \wedge dZ_m\right) = \Big(Lg - \sum_{k=1}^{m} M_k(g) LZ_k \Big) dt \wedge dZ_1 \wedge \cdots \wedge dZ$$

Let $Z = (Z_1, \ldots, Z_m)$. For $\xi \in \mathbb{R}^m$, $s \in \mathbb{R}^m$,

let $E(s,\xi,x,t) = i\xi\cdot(s - Z(x,t)) - |\xi|\,[s - Z(x,t)]^2$.

Let B be a small ball centered at $x_0 = 0$, $\psi \in C_0^\infty(B)$, $\psi \equiv 1$ near $x_0 = 0$. Apply $(**)$ to $g(s,\xi,x,t) = \psi(x) f(x,t)\, e^{E(s,\xi,x,t)}$, where (s,ξ) are parameters. We get: with $dZ = dZ_1 \wedge \cdots \wedge dZ$

$$(**)'\quad d(g\, dZ) = \Big\{ L(\psi f) + (\psi f) LE - \sum_{k=1}^{m} \big(M_k(\psi f) + \psi f (M_k E) \big) LZ_k \Big\} e^{E}\, dt \wedge dZ$$

By Stoke's thm, for $t_1 > 0$ small:

$$\int_B g(s,\xi,x,0)\, dx = \int_B g(s,\xi,x,t_1)\, d_x Z(x,t_1) + \int_0^{t_1}\!\!\int_B d(g\, dZ) \quad (***)$$

We'll estimate the two integrals on the right in $(***)$:

$Z = (z_1, \dots, z_m) = x + t\varphi(x,t)$, $\varphi = \varphi_1 + i\varphi_2$.

Since the Z_j are approx solns of L,

$$\varphi + t\varphi_t + (I + t\varphi_x)\cdot a = O(t^k) \quad \forall k.$$

$$\Rightarrow \quad \varphi(x,0) = -a(x,0).$$

$$\Rightarrow \quad \langle \operatorname{Im}\varphi(0,0), \xi^0\rangle < 0. \quad \therefore \text{ for } x \in \overline{B},$$

$0 \le t \le t_1$ (after shrinking B), $\langle \operatorname{Im}\varphi(x,t), \xi^0\rangle \le -c < 0$. $(*')$

Note that

$$\operatorname{Re} E(s,\xi,x,t) = t\,\xi\cdot\varphi_2(x,t) - |\xi|\left[(s - x - t\varphi_1)^2 - t^2\varphi_2(x,t)^2\right]$$

Using $(*')$ and homogeneity in ξ, $\exists\, c_1 > 0 \ni$

$$\operatorname{Re} E(s,\xi,x,t) \le -c_1|\xi|t \quad \text{for } x \in \overline{B},\ 0 \le t \le t_1,$$

$s \in \mathbb{R}^m$ and ξ in a conic nbhd of ξ^0. Hence,

in $(**)$

$$\left|\int_B g(s,\xi,x,t_1)\, d_x Z(x,t_1)\right| \le e^{-c_2|\xi|} \quad \text{for some } c_2 > 0,$$

$s \in \mathbb{R}^m$, $\xi \in \Gamma$. To estimate $\int_0^{t_1}\int_B d(g\,dZ)$, use

$(**)'$: First consider the term $L(\psi f)\, e^E$.

For any k, $\quad |\psi(Lf)e^E| \le C_k t^k e^{-c_1|\xi|t} \le \dfrac{C_k'}{|\xi|^k}$

We get exp decay from $(L\psi)f\,e^E$ for s small.

$\therefore \displaystyle\int_B\int_0^{t_1} L(\psi f)\, e^E\, dt \wedge dZ$ decays rapidly in ξ.

The term $(\psi f)\, LE\, e^E$ is estimated using the fact that for any k, $|LE| \le c_k t^k |\xi|$ and $|e^E| \le e^{-c_1 t |\xi|}$. Thus

$$\int_B\int_0^{t_1} (\psi f)\, LE\, e^E\, dt \wedge dZ \quad \text{decays rapidly in } \xi.$$

The other terms are estimate in the same way. Thus

$$\int_B\int_0^{t_1} d(g\,dZ) \quad \text{has a rapid decay in } \xi.$$

From $(**)$, we conclude:

$$F(s,\xi) = \int_B e^{i\xi\cdot(s-x) - |\xi||s-x|^2}\, \psi(x)\, f(x,0)\, dx$$

decays rapidly for $|s| \le \delta$ and $\xi \in \Gamma =$ conic nbhd of ξ^0.

$\therefore (0,\xi^0) \notin WF(f(x,0))$.

Corollary: If $Lf = 0$ and $LZ_j = 0$, the proof shows that $(0,\xi^0) \notin WF_a(f(x,0))$.

Consider now $L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x,t)\frac{\partial}{\partial x_j}$, $a_j \in C^1$.

For each $\theta \in [0, 2\pi)$, let

$$L^\theta = \frac{\partial}{\partial s} - e^{-i\theta} L .$$

Spose for each θ, $\exists$ C^1 fns

$$\varphi_1^\theta(x,t,s), \dots, \varphi_m^\theta(x,t,s) \ni$$

$Z_j^\theta(x,t,s) = x_j + s\,\varphi_j^\theta(x,t,s)$ $(1 \le j \le m)$ are approximate solns of L^θ, $L^\theta Z_j^\theta(x,t,s) = O(s^k)$ $\forall k$.

Define also $\varphi_{m+1}^\theta(x,t,s) = e^{-i\theta}$ and $Z_{m+1}(x,t,s) = t + e^{-i\theta}s$.

and $Z_{m+1}(x,t,0) = t$.

Note that $L^\theta Z_{m+1}^\theta = 0$

Write $\varphi^\theta = (\varphi_1^\theta, \dots, \varphi_{m+1}^\theta)$, $Z^\theta = (Z_1^\theta, \dots, Z_{m+1}^\theta)$.

Then $Z_s^\theta(0,0,0) = \varphi^\theta(0,0,0) = -\begin{pmatrix} e^{-i\theta} a(0,0) \\ e^{-i\theta} \end{pmatrix} = e^{-i\theta}\begin{pmatrix} \varphi(0,0) \\ -1 \end{pmatrix}$

and

$$\langle (\xi,\tau), \operatorname{Im}\varphi^\theta(0,0,0)\rangle$$

$$= \langle (\xi,\tau), (\operatorname{Im}\varphi(0,0)\cos\theta - \operatorname{Re}\varphi(0,0)\sin\theta, \sin\theta)\rangle$$

$$= \langle \xi, \operatorname{Im}\varphi(0,0)\rangle \cos\theta + (\tau - \langle \xi, \operatorname{Re}\varphi(0,0)\rangle)\sin\theta .$$

Hence, the condition that

$$\langle (\xi, \tau), \operatorname{Im} \varphi^{\theta}(0,0,0) \rangle \neq 0 \quad \text{for some } \theta \in [0, 2\pi)$$

$$\Longleftrightarrow \quad (0, 0, \xi, \tau) \notin \operatorname{Char} L.$$

We have proved:

<u>Theorem</u>: Let $Lh(x,t) = 0$ ~~for $0 < t < \delta$~~. Then

$$WF(h) \subseteq \operatorname{Char} L.$$