Stellar Structure

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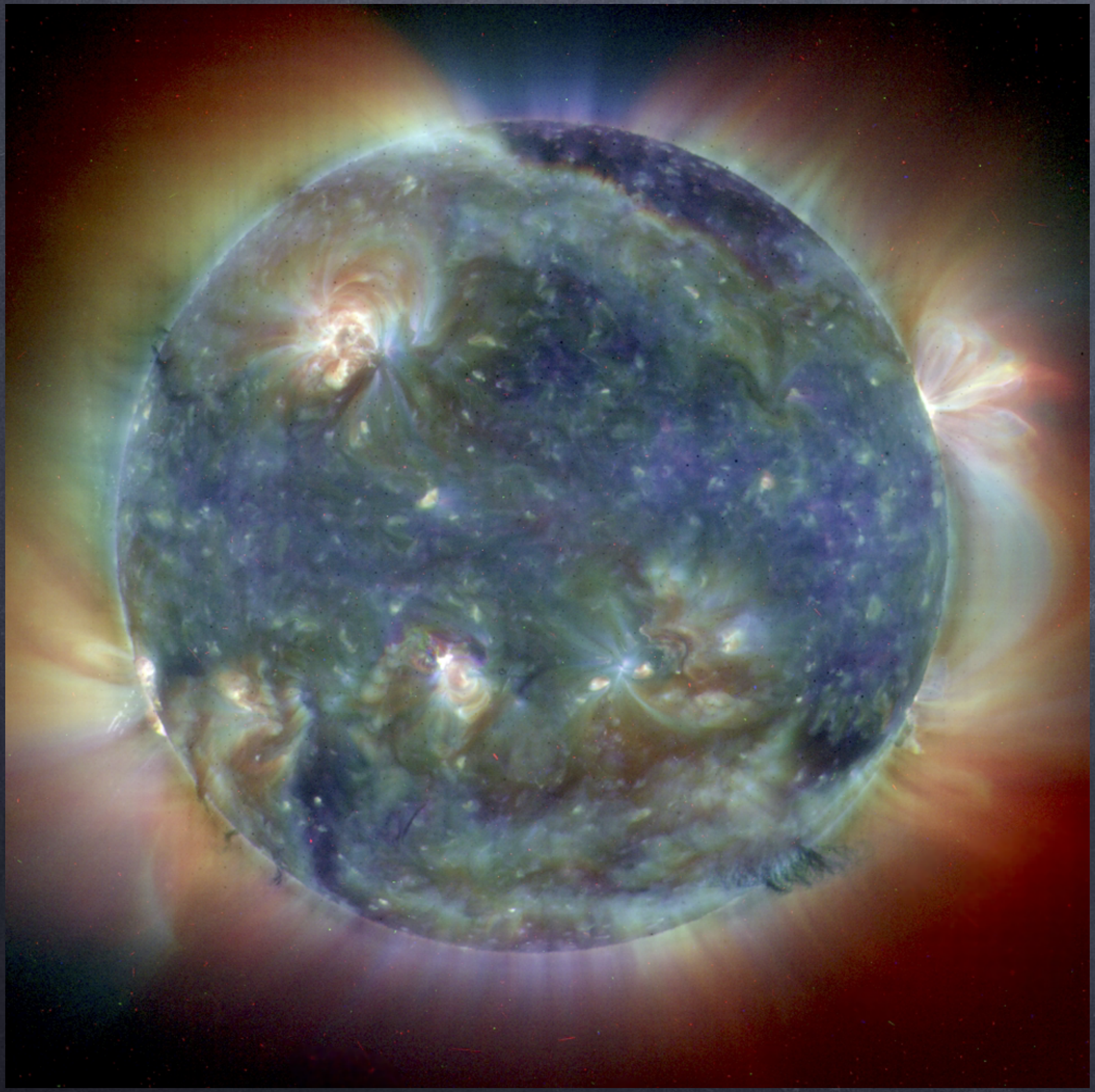
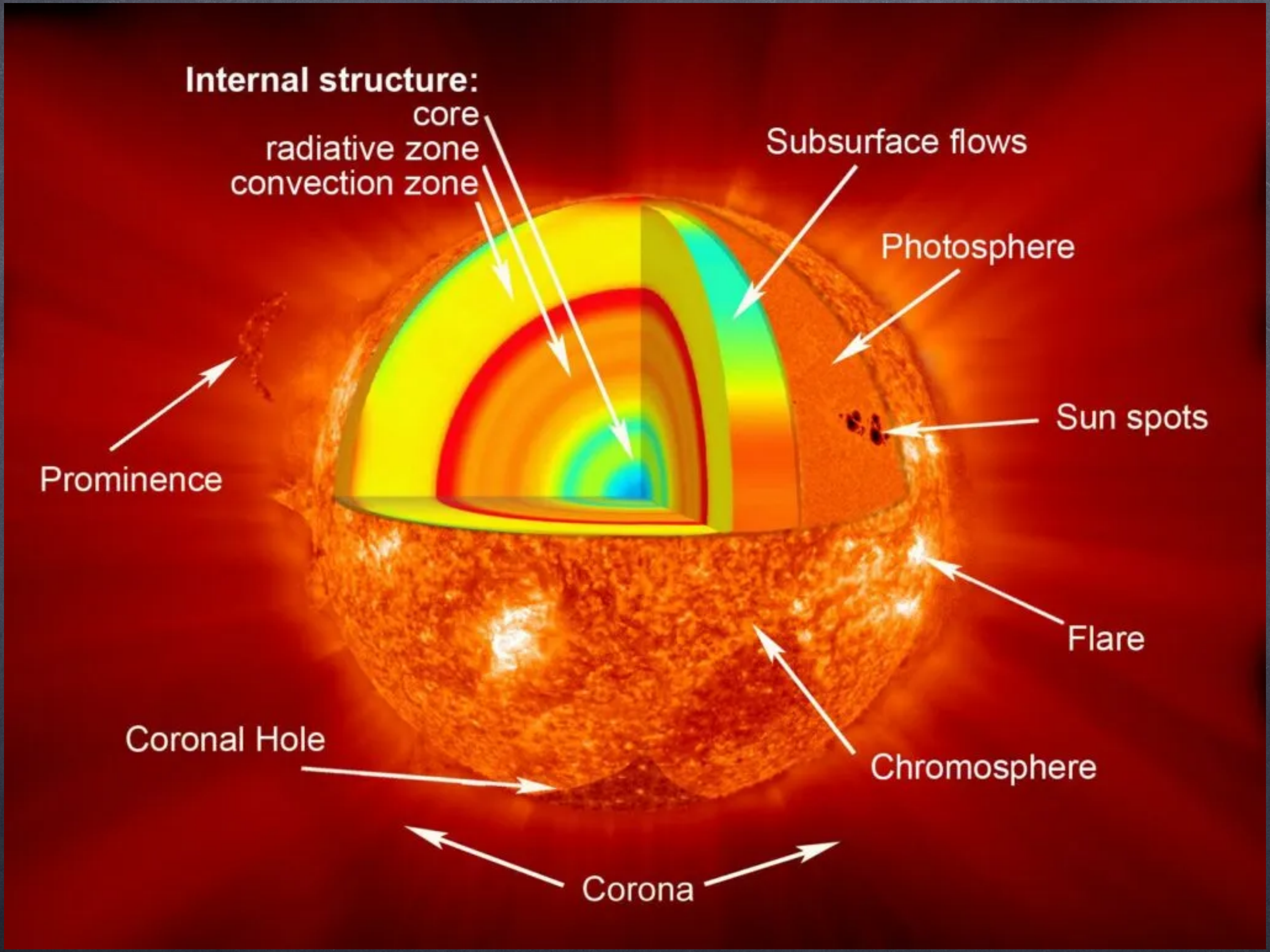
The Sun

SOHO (Solar and Heliospheric Observatory) spacecraft image of the Sun in May 1998



- Composite image of the Sun made by combining observations in three wavelengths (171, 195 and 284 angstroms) of ultraviolet (UV) light to reveal solar features unique to each wavelength, which are colour-coded red, yellow and blue

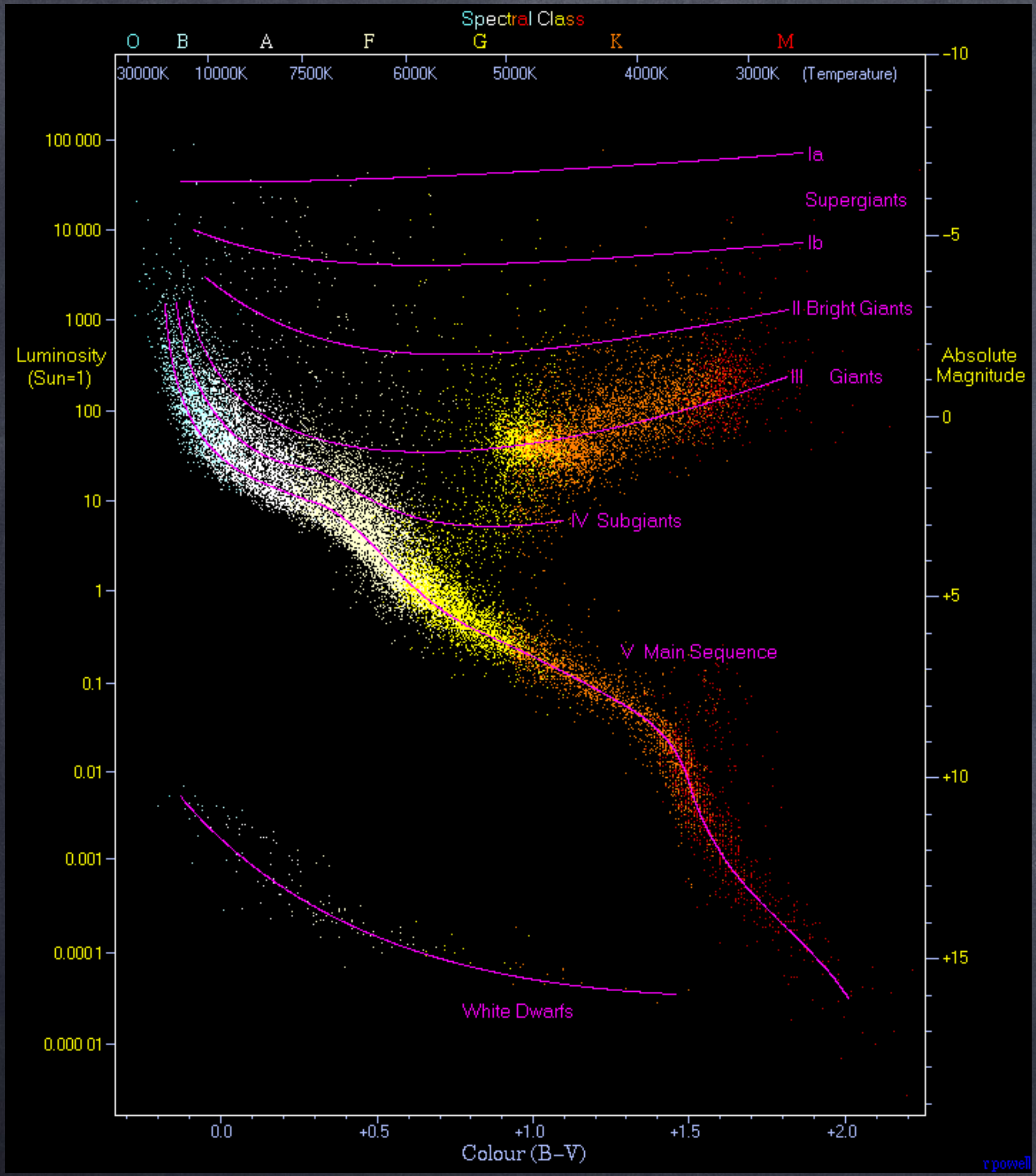
Structure of the Sun



The Pleiades

- "Open star cluster" containing young, hot, massive (B-type) stars

Hertzsprung-Russell diagram of nearby Stars



22000 stars from the Hipparcos Catalogue and 1000 stars from the Gliese Catalogue

- Brightness & Colour are the most easily observed qualities of a star
- H-R diagram also called Colour-Magnitude diagram
- Parallax gives distance to a star –> Luminosity $L_\star$
- Stars radiate like blackbodies, Colour –> effective Temperature $T_{\rm eff}$
- HR diagram –> Plot of $L_\star$ vs $T_{\rm eff}$

Stars on the Main Sequence

- A star is a nearly spherical ball of gas, composed of H, He and other elements (called "metals"), that exists in a near steady-state for a "long time", while emitting light (between IR and UV) from its surface
- The steady-state is due to hydrostatic equilibrium, which is a balance between gravity and pressure (of gas + radiation)
- Density and temperature (hence pressure) increase towards the centre of the star
- Nuclear reactions in the core convert H to He which releases energy in the form of radiation (and neutrinos which escape)
- The radiation diffuses outward while heating the gas
- The star exists in a near steady-state until the H in the core is exhausted

Plan of Lectures

I. Hydrostatic Equilibrium

II. Nuclear Energy Generation

III. Energy Transport

IIIa. Radiative Energy Transport; Opacity

IIIb. Convective Energy Transport*

IV. Main Sequence

Some physical quantities

Some basic observables:

$M_\star$ = Mass of the star $\qquad M_\odot \simeq 2 \times 10^{30}\,\mathrm{kg}$

R = Radius of the star $\qquad R_\odot \simeq 700,000\,\mathrm{km}$

$L_\star$ = Luminosity of the star (Energy/time) $\qquad L_\odot \simeq 3.8 \times 10^{26}\,\mathrm{W}$

Some physical quantities:

$\rho(r)$ = Mass density at radius r inside the star

$M(r) = \int_0^r \rho(r') 4\pi r'^2 \mathrm{d}r'$ = Mass enclosed within r

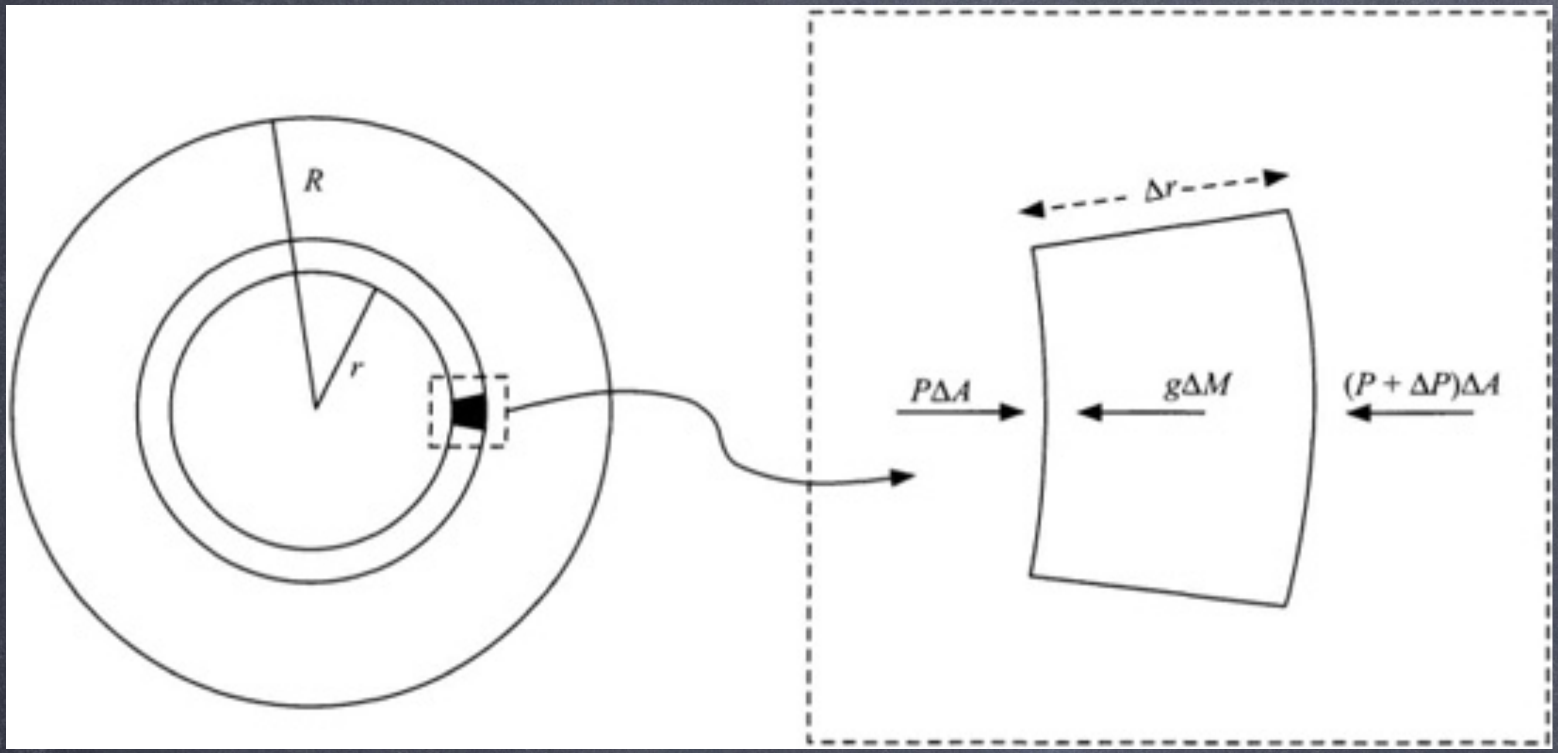
$M(R) = M_\star$

$T(r)$ = Temperature at r

$P(r) = P_{\mathrm{gas}}(r) + P_{\mathrm{rad}}(r)$ = Pressure at r

I. Hydrostatic Equilibrium

- Balance between gravity and pressure (of gas + radiation)



$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g(r)\rho(r)$$

where

$$g(r) = \frac{GM(r)}{r^2}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho(r) \qquad (1)$$

where

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\rho(r) \qquad (2)$$

- Equation (1) and (2) can be used to estimate the central pressure:

Since $P(R) = 0$, we estimate $\frac{\mathrm{d}P}{\mathrm{d}r} \approx -\frac{P_c}{R}$

We also set $M(r) = M_\star$ and $\rho(r) = M_\star / \left(\frac{4\pi}{3} R^3\right)$

Then

$$P_c \approx \frac{3}{4\pi} \frac{G M_\star^2}{R^4}$$

Problem: Estimate P_c for the Sun. This is not a good estimate!
But the important point are the scaling relations:

$$P_c \propto \frac{M_\star^2}{R^4}$$

$$\overline{\rho} \propto \frac{M_\star}{R^3}$$

Dynamical (or free-fall) timescale

Consider a constant density star of mass $M_\star$ and radius R in hydrostatic equilibrium

Imagine that the pressure suddenly vanishes! Since the inward force of gravity is unbalanced, the star will collapse. The time taken to collapse to a point is called the dynamical time.

The problem can be solved exactly. But we only need an estimate:

The surface gravity is $g(R) = \dfrac{GM_\star}{R^2}$

$$t_{\rm dyn} \approx \left(\frac{R}{g(R)}\right)^{1/2} \approx \frac{1}{2}\left(\frac{1}{G\overline{\rho}}\right)^{1/2} \qquad (3)$$

This is also the time scale for readjustment when a star in hydrostatic equilibrium is slightly perturbed

Problem: Estimate for the Sun $t_{\rm dyn} \approx 0.5$ hour

- The Gravitational Potential Energy W is defined as the negative of the work done to remove the mass of the star to infinity, peeling it shell by shell from the outside in. Then

$$W = -4\pi G \int_0^R r M(r)\rho(r)\,\mathrm{d}r \qquad (4)$$

Problem: For a constant density sphere, show that $W = -\frac{3}{5}\frac{GM_\star^2}{R}$

Using (1) to replace the product $M(r)\rho(r)$, and manipulating the integral, we can derive

$$W = -3\int_0^R P(r) 4\pi r^2 \mathrm{d}r = -3\overline{P}V \qquad (5)$$

where $\overline{P}$ is the volume-averaged pressure and V is the volume of the star

- The Pressure $P(r) = P_{\text{gas}}(r) + P_{\text{rad}}(r) \simeq P_{\text{gas}}(r)$ because radiation pressure is usually small. So we use

$$P = nk_B T$$

For a monoatomic gas, the kinetic energy per volume is $3P/2$, so the Kinetic Energy (or Thermal Energy) of the star is

$$K = \frac{3}{2}\overline{P}V \tag{6}$$

From (5) and (6) we obtain

$$W = -2K \tag{7}$$

Since the Total Energy $E = K + W$, we have

$$E = -K = W/2 \tag{7'}$$

Equation (7′) is called the Virial Theorem

Kelvin-Helmholtz (or thermal) timescale

As a star radiates it loses energy, while remaining in quasi-hydrostatic (or virial) equilibrium

If it did not have internal energy sources (like nuclear fusion), it can shine at the rate $L_\star$ for a maximum time of order

$$t_{KH} = \frac{|E|}{L_\star} \tag{8}$$

Assuming a constant density star, the virial theorem gives

$$E = W/2 = -\frac{3}{10}\frac{GM_\star^2}{R}$$

which implies

$$t_{KH} \approx \frac{3}{10}\frac{GM_\star^2}{RL_\star} \tag{8'}$$

Problem: Estimate for the Sun that $t_{KH} \approx 10^7$ yr

- Expressing P as a function of ρ, T

m_0 = atomic mass unit

μ_i = Molecular weight of nuclei of type i = H, He, etc

Z_i = Atomic number of ...

$$P = nk_BT = \left(n_e + \sum_i n_i\right) k_BT$$

Define Mass Fractions of different nuclei $X_i = \dfrac{\rho_i}{\rho} = \dfrac{n_i \mu_i m_0}{\rho}$

For fully ionised gas $n_e = \sum_i Z_i n_i$

Then $$P = \frac{\rho k_B T}{\mu m_0} \qquad (9)$$

with $$\frac{1}{\mu} = \sum_i \frac{X_i\left(1 + Z_i\right)}{\mu_i}$$; μ = mean molecular weight

Use notation $X_H = X\,, \quad X_{He} = Y\,, \quad \sum_{i=\text{metals}} X_i = Z$

By definition $X + Y + Z = 1$

Have $Z_H = 1\,, \mu_H = 1\,; \quad Z_{He} = 2\,, \mu_{He} = 4\,; \quad \frac{1+Z_i}{\mu_i} \simeq \frac{Z_i}{\mu_i} \simeq \frac{1}{2}$ for metals

Then

$$\frac{1}{\mu} \simeq 2X + \frac{3}{4}Y + \frac{1}{2}Z \qquad (10)$$

For Solar composition gas:

$$X \simeq 0.7\,, \quad Y \simeq 0.28\,, \quad Z \simeq 0.02$$

(10')

So $\mu_\odot \simeq 0.62$

II. Nuclear Energy Generation

- Nuclear reactions (fusion or fission) proceed in the direction of increasing the binding energy per nucleon
- In main-sequence stars, 4 protons fuse to form a ^4He nucleus. This happens through the p-p chain (in stars like the Sun), or the CNO-cycle (in higher mass stars)
- Mass of 4 protons = $4 \times 1.0079 \text{ a.m.u.} = 4.0316 \text{ a.m.u.}$
- Mass of a ^4He nucleus = 4.0026 a.m.u.
- Mass deficit $\Delta m = (4.0316 - 4.0026) \text{ a.m.u.} = 0.029 \text{ a.m.u.}$
- Energy released $\Delta E = \Delta m\, c^2 \simeq 27 \text{ MeV}$
- About 0.7% of the rest energy of 4 protons has been released

The pp chain and CNO cycle

There are two chief routes by which hydrogen can fuse into helium. One is the proton–proton chain,[21] of which the simplest version is[22]

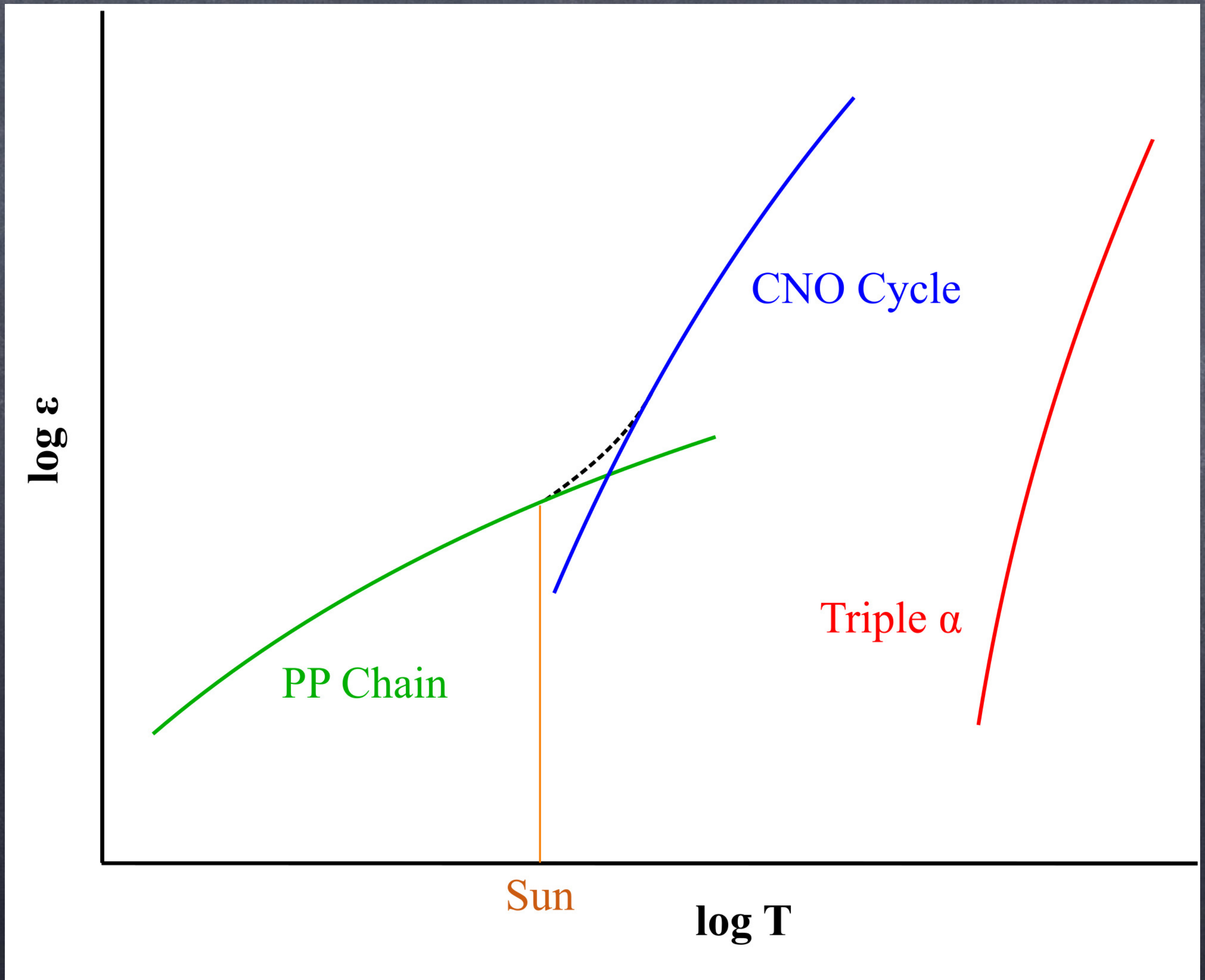
$$\begin{aligned}
\text{I}: &\quad {}^{1}\mathrm{H} + {}^{1}\mathrm{H} \to {}^{2}\mathrm{H} + e^{+} + \nu_{e} + 1.18\ \mathrm{MeV} \\
\text{II}: &\quad {}^{1}\mathrm{H} + {}^{2}\mathrm{H} \to {}^{3}\mathrm{He} + \gamma + 5.49\ \mathrm{MeV} \\
\text{III}: &\quad {}^{3}\mathrm{He} + {}^{3}\mathrm{He} \to {}^{4}\mathrm{He} + {}^{1}\mathrm{H} + {}^{1}\mathrm{H} + 12.85\ \mathrm{MeV}.
\end{aligned} \tag{1.5.2}$$

The other route is the CNO cycle,[23] which in its simplest variant is

$$\begin{aligned}
\text{i}: &\quad {}^{1}\mathrm{H} + {}^{12}\mathrm{C} \to {}^{13}\mathrm{N} + \gamma + 1.95\ \mathrm{MeV} \\
\text{ii}: &\quad {}^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^{+} + \nu_{e} + 1.50\ \mathrm{MeV} \\
\text{iii}: &\quad {}^{1}\mathrm{H} + {}^{13}\mathrm{C} \to {}^{14}\mathrm{N} + \gamma + 7.54\ \mathrm{MeV} \\
\text{iv}: &\quad {}^{1}\mathrm{H} + {}^{14}\mathrm{N} \to {}^{15}\mathrm{O} + \gamma + 7.35\ \mathrm{MeV} \\
\text{v}: &\quad {}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^{+} + \nu_{e} + 1.73\ \mathrm{MeV} \\
\text{vi}: &\quad {}^{1}\mathrm{H} + {}^{15}\mathrm{N} \to {}^{12}\mathrm{C} + {}^{4}\mathrm{He} + 4.96\ \mathrm{MeV},
\end{aligned} \tag{1.5.3}$$

Each step involves either an Electromagnetic or Weak Interaction

- The nuclear reactions involved in either the p-p chain or the CNO cycle involve a series of <u>2-body encounters between positively charged nuclei</u>
- Since <u>like charges repel</u>, quantum tunnelling helps to penetrate the "Coulomb barrier". Also, the process is more likely at higher temperatures



- The probability of encounters between nuclei is higher at higher densities

- Higher temperatures and densities imply that the nuclear reactions are confined to the core of a star

  - Energy released when $0.1\,M_\odot$ of H fuses over the lifetime of the Sun is

$$E_{\rm tot} \;\approx\; 7 \times 10^{-4} \times M_\odot c^2 \;\approx\; 10^{44}\ {\rm J}$$

  - The Solar luminosity is

$$L_\odot \;\simeq\; 3.83 \times 10^{26}\ {\rm W}$$

  - If the Sun continued radiating with this luminosity, the main-sequence lifetime (also called nuclear timescale) is

$$t_{\rm nuc} \;\approx\; \frac{E_{\rm tot}}{L_\odot} \;\simeq\; 2.6 \times 10^{17}\ {\rm s} \;\approx\; 10^{10}\ {\rm yr} \tag{11}$$

Let $L(r)$ = luminosity through the spherical surface of radius r

Let $\varepsilon(r)$ = rate of release of nuclear energy per unit mass

Then the luminosity added to $L(r)$ by a spherical shell of radius r and thickness $\mathrm{d}r$ is $\mathrm{d}L = \rho\varepsilon\, 4\pi r^2 \mathrm{d}r$. Therefore

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho(r)\varepsilon(r) \tag{12}$$

ε is a function of ρ, T and chemical composition. It is $\propto \rho$ and is a very sensitive function of T . We write

$$\varepsilon = \mathrm{constant} \times \rho T^{\nu} \tag{13a}$$

where

$$\nu \approx \begin{cases} 5 & \text{(p-p chain)} \\ 15 & \text{(CNO cycle)} \end{cases} \tag{13b}$$

III. Energy Transport

- The Energy generated in the stellar core by nuclear reactions is released in the form of high-frequency (X-ray) photons. These travel outward, while interacting and exchanging energy with the stellar material.

- The total Energy flux density at r is

$$F(r) = \frac{L(r)}{4\pi r^2} \tag{14}$$

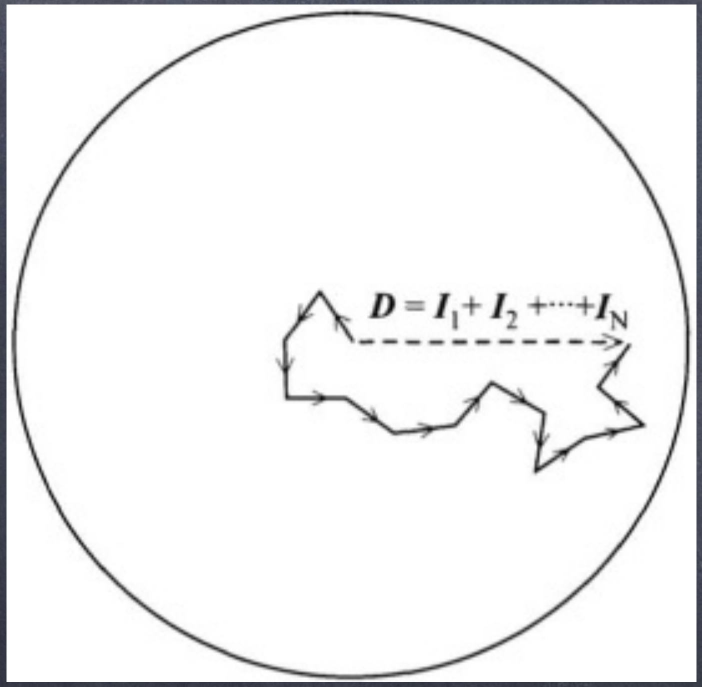
$F(r)$ has units Energy/area/time

Energy can be transported by radiation (photons) and convection (bulk motion of gas)

- Outside the core $\varepsilon(r) \simeq 0$
Then, (12) implies that $L(r) \simeq L_\star = \mathrm{constant}$ outside the core

IIIa. Radiative Energy Transport

- The photons travel toward the surface, while interacting with the gas through absorption, re-emission and scattering processes



We can think of a photon as executing a random walk*. This is a diffusive process, with mean free path

$$\ell = \frac{1}{n_0 \sigma}$$

where

n_0 = effective number density of gas particles interacting with the photon

σ = photon cross-section

*while also decreasing in frequency as it travels from hotter to cooler regions

We write $n_0\sigma = \rho\kappa$ where κ = Opacity of the stellar material (cross-section/mass) that has to be calculated. Then $\ell = 1/(\rho\,\kappa)$

In a diffusive process

$$F(r) = -D\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}r}$$

where

$$\mathcal{E}(r) = aT^4$$ = Energy density of radiation (Energy/volume)

$$D(r) = \frac{\ell c}{3} = \frac{c}{3\rho\kappa}$$ = Diffusion coefficient

Work out

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}r} = \frac{\mathrm{d}aT^4}{\mathrm{d}r} = 4aT^3\frac{\mathrm{d}T}{\mathrm{d}r}$$

Then the Energy flux density is given by Fourier's law of heat conduction:

$$F(r) = -k_{\rm cond}\frac{{\rm d}T}{{\rm d}r}$$

where the conductivity is due to photon diffusion:

$$k_{\rm cond} = \frac{4acT^3}{3\rho\kappa}$$

Using equation (14) to relate the flux to luminosity, we obtain the equation governing the temperature gradient:

$$\frac{{\rm d}T}{{\rm d}r} = -\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2} \tag{15}$$

All the key information is contained in the Opacity function $\kappa(r)$

- Opacity κ is a function of ρ, T and chemical composition. It can be approximated over some ranges as a power-law:

$$\kappa = \text{constant} \times \rho^{\alpha} T^{\beta} \tag{16a}$$

Thomson scattering: elastic scattering of a photon by an electron

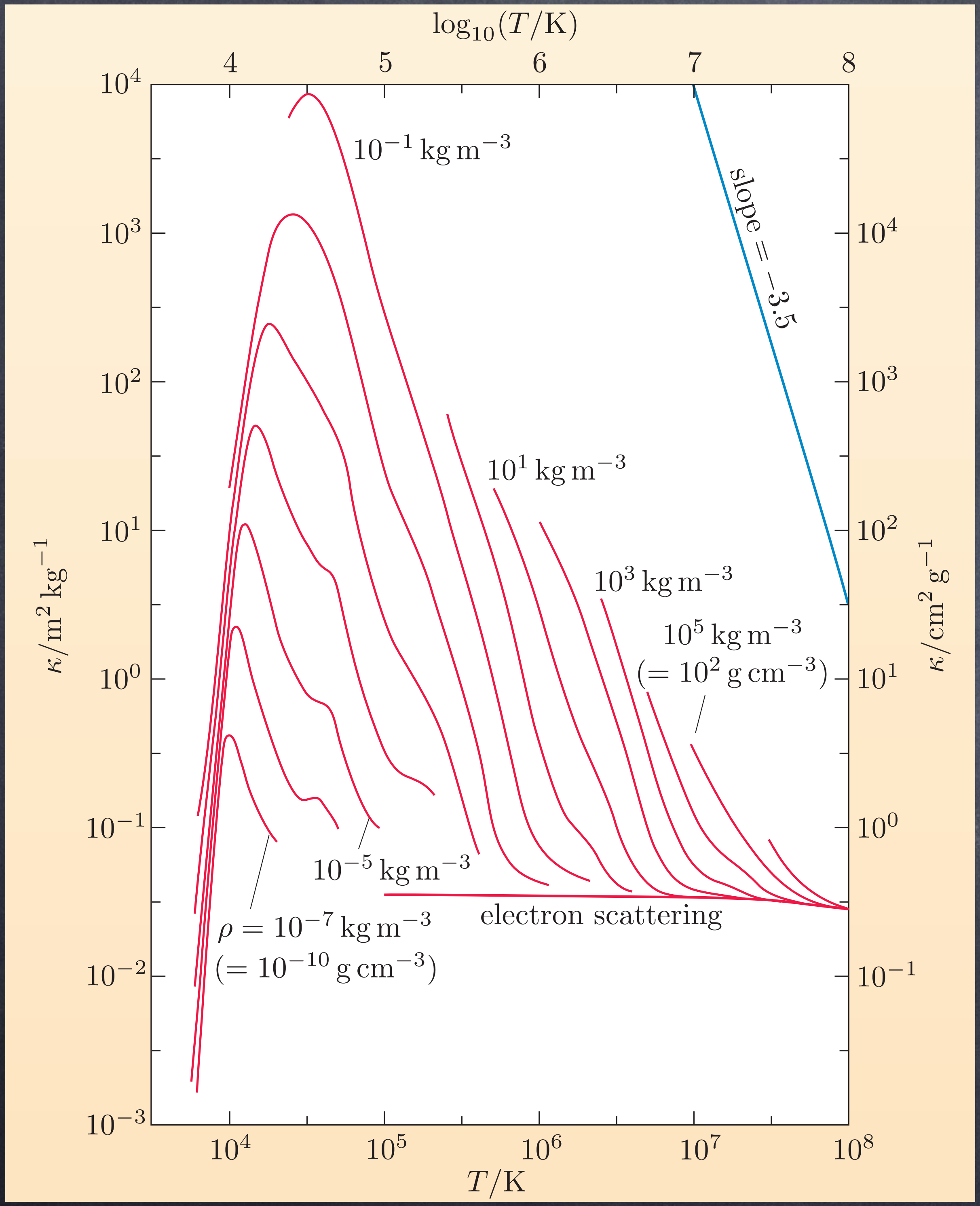
$$\alpha = 0\,, \qquad \beta = 0 \tag{16b}$$

Free-Free absorption: absorption of a photon by an electron in the presence of a positively charged ion or nucleus

$$\alpha = 1\,, \qquad \beta = -7/2 \qquad \text{(Kramers opacity)} \tag{16c}$$

Other processes like Bound-Free and Bound-Bound absorption involve interactions between photons and bound electrons

Opacity of Solar composition material



For $T < 3000K$ above processes don't contribute much to the Opacity because there are not enough free electrons

- Kramers Opacity: good approximation for a range of ρ, T

- Thomson scattering dominates at high T

IIIb. Convective Energy Transport

From (15) we see that, for radiation to transport energy, the larger the luminosity $L(r)$, the larger must $|\mathrm{d}T/\mathrm{d}r|$ be

The Schwarzschild Instability Criterion states that the hydrostatic equilibrium is unstable to (radial) convective motions when

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right| > \frac{g(r)}{c_p} \qquad (17)$$

where $g(r) = \dfrac{GM(r)}{r^2}$ = gravitational acceleration

For a given $L_\star$, there could be regions in a star — like the outer 30% of the Sun — where $|\mathrm{d}T/\mathrm{d}r|$ as calculated from (15) is larger than $g(r)/c_p$.

Then energy is transported efficiently by convection, and $\mathrm{d}T/\mathrm{d}r \simeq -g(r)/c_p$

IV. Main Sequence

- Equations of Stellar Structure

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho(r)$$ where $$P = \frac{\rho k_B T}{\mu m_0} + \frac{aT^4}{3}$$

with $$\frac{1}{\mu} \simeq 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho(r)$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho(r)\varepsilon(r)$$ where $$\varepsilon = \mathrm{constant} \times \rho T^\nu$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2}$$ where $$\kappa = \mathrm{constant} \times \rho^\alpha T^\beta$$

Boundary Conditions

$M(0) = 0$ $\quad L(0) = 0$

$T(R) = 0$ $\quad \rho(R) = 0$

Scaling Relations (for fixed chemical composition)

$$\frac{P}{R} \propto \frac{M\rho}{R^2}$$ where $$P \propto \rho T$$

(ignoring radiation pressure)

$$\frac{M}{R} \propto R^2\rho$$

$$\frac{L}{R} \propto R^2\rho\,\varepsilon$$ where $$\varepsilon \propto \rho T^\nu$$

$$\frac{T}{R} \propto \frac{\rho\kappa}{T^3}\frac{L}{R^2}$$ where $$\kappa \propto \rho^\alpha T^\beta$$

Any quantity $Q \propto M^\eta$ where $Q = R, T, L,$ etc

Should also choose α, β, ν appropriately

Important examples

$$R \propto M^{\eta_R} \qquad \text{where} \qquad \eta_R = \frac{\nu+\alpha+\beta-1}{3+\nu+3\alpha+\beta}$$

$$L \propto M^{\eta_L} \qquad \text{where} \qquad \eta_L = (2+\nu)-(3+\nu)\eta_R$$

*Main Sequence lifetime = $t_{\rm nuc} \propto \dfrac{M}{L} \propto M^{(1-\eta_L)}$

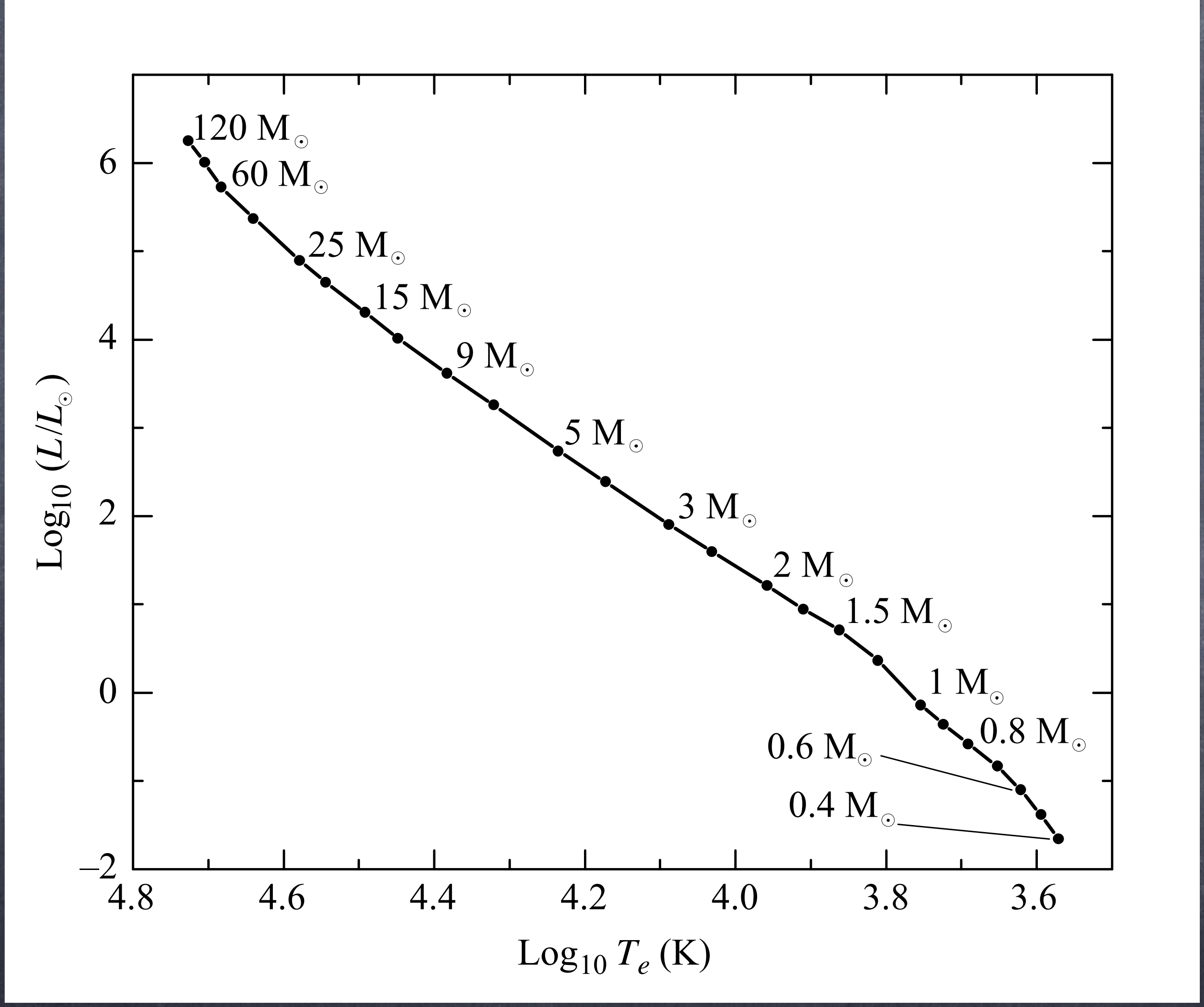
*Since stars radiate like blackbodies $L = 4\pi R^2\,\sigma T_{\rm eff}^4$

$$T_{\rm eff} \propto \frac{L^{1/4}}{R^{1/2}} \propto M^{\eta_e} \qquad \text{where} \qquad \eta_e = \frac{\eta_L}{4} - \frac{\eta_R}{2}$$

Problem: Work out $\eta_R\,,\,\eta_L\,,\,\eta_e$ for the case of the CNO cycle $\nu \approx 15$ and Kramers opacity $\alpha = 1\,,\beta = -7/2$ (valid for $2M_\odot < M_\star < 20M_\odot$)

Theoretical HR diagram (ZAMS)

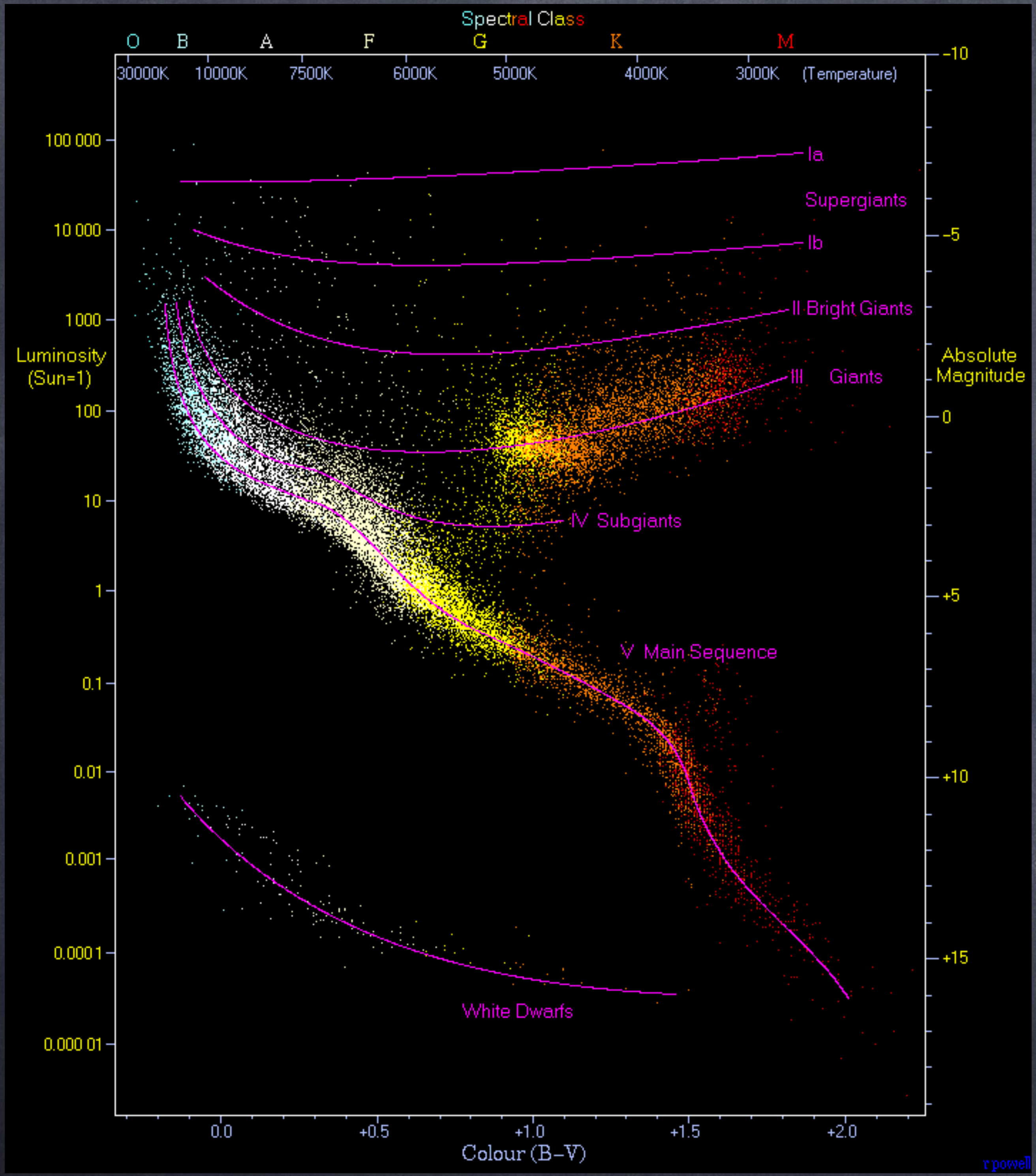
All physical properties of a star are decided by its Mass and Chemical composition when it is born



Problem: Work out $L \propto T_{\rm eff}^{(\eta_L/\eta_e)}$

Compare with the above figure for the mass range $2M_\odot < M_\star < 20M_\odot$

Hertzsprung-Russell diagram of nearby Stars



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