Mathematics of Waves and Coastal Hazards

From Simple Equations to Numerical Ideas

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mathematics + computation + coastal engineering

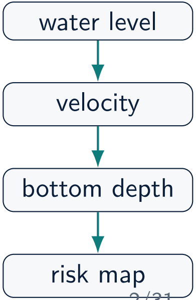
Why this workshop?

Main idea

Mathematics can transform a dangerous natural phenomenon into a model that can be studied, simulated, and used for decisions.

Observation $\rightarrow$ Model $\rightarrow$ Computation $\rightarrow$ Risk

- Waves are visible, physical, and intuitive.
- The equations are simple to start, deep when developed.
- They connect analysis, numerics, geology, and engineering.



Plan for one hour

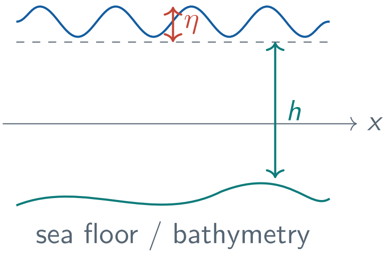
1. **10 min** - What is a wave? What do we measure?
2. **12 min** - First equations: conservation and shallow water.
3. **12 min** - Tsunamis, geology, and the sea floor.
4. **10 min** - Surface tension and dispersion.
5. **10 min** - Simple computing: from equations to algorithms.
6. **6 min** - Discussion: what can students do next?

Goal: understand the **logic of modeling**, not memorize formulas.

A wave: the minimum vocabulary

- x: horizontal position.
- t: time.
- $\eta(x,t)$: free-surface elevation.
- $h(x)$: still-water depth.
- $u(x,t)$: horizontal velocity.

total water height:
$$H(x,t) = h(x) + \eta(x,t)$$



First mental model: a traveling wave

$$\boxed{\eta(x, t) = A\cos(kx - \omega t)}$$

- A: amplitude.
- $k = 2\pi/\lambda$: wave number.
- $\omega = 2\pi/T$: frequency.
- $c = \omega/k$: phase speed.

Question to students

If A doubles, is the wave only twice as high, or can the dynamics change qualitatively?

Key word

linearity vs nonlinearity

Conservation: the language of modeling

A model usually begins with a balance law:

$$\boxed{\text{rate of change} + \text{flux through boundaries} = \text{source}}$$

For water waves, two balances are fundamental:

Mass

Water is not created or destroyed.

$$\partial_t H + \partial_x (Hu) = 0.$$

Momentum

Velocity changes due to pressure, gravity, bottom effects, and forces.

$$\partial_t u + u\partial_x u + g\partial_x \eta = \cdots$$

Shallow-water equations: a first coastal model

When wavelength λ is much larger than depth h, a useful model is:

$$\boxed{\begin{aligned} \eta_t + \big((h+\eta)u\big)_x &= 0, \\ u_t + uu_x + g\eta_x &= 0. \end{aligned}}$$

- Nonlinear: the terms $(h+\eta)u$ and uu_x matter for large waves.
- Hyperbolic: information travels with finite speed.
- Very useful for tsunami propagation and inundation models.

A powerful estimate: tsunami speed

For long waves in the ocean,

$$\boxed{c = \sqrt{gh}}$$

- $g \approx 9.81\,\mathrm{m/s^2}$.
- If $h = 4000\,\mathrm{m}$, then

$$c \approx 198\,\mathrm{m/s} \approx 713\,\mathrm{km/h}.$$

Interactive calculation

How long would a tsunami take to cross 1000 km of ocean with average depth 4000 m?

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

The role of geology: the bottom is part of the equation

The sea floor is not just a boundary. It controls wave speed, focusing, and run-up.

$$\boxed{h = h(x, y)}$$

- Continental shelf: waves slow down and grow.
- Submarine canyons: energy may focus.
- Earthquake deformation: the bottom moves and generates the wave.

Mathematical message

Changing the geology means changing coefficients in the PDE.

$$\boxed{c(x, y) = \sqrt{gh(x, y)}}$$

From earthquake to tsunami: a source term

A tsunami can be generated when the sea floor moves vertically.

$$\boxed{\text{bottom displacement: } b(x, y, t)}$$

A simple way to express the idea is:

$$\eta_t + \nabla \cdot (H\mathbf{u}) = -b_t.$$

Interpretation

If the bottom rises quickly, the water surface is pushed upward. The source is not a wave at first; it becomes a wave through propagation.

Shoaling: why waves grow near shore

As depth decreases, long waves slow down:

$$\boxed{c = \sqrt{gh} \text{ decreases as } h \text{ decreases}}$$

If energy flux is approximately conserved, wave height can increase.

Qualitative chain

deeper water $\rightarrow$ faster wave
shallower water $\rightarrow$ slower wave
wave energy compresses $\rightarrow$ larger amplitude

Risk consequence

Small offshore amplitude can become dangerous near the coast.

Dispersion: not all waves travel at the same speed

A dispersive model allows speed to depend on wavelength.

$$\omega = \omega(k), \qquad c(k) = \frac{\omega(k)}{k}$$

- Long waves and short waves may separate.
- The shape of a wave packet changes with time.
- Dispersive models are central in Boussinesq-type equations.

Simple question

If two wave components travel at different speeds, can a wave remain the same shape forever?

Surface tension: small scale, big mathematical effect

For capillary-gravity waves, a classical linear dispersion relation is:

$$\boxed{\omega^2 = \left(gk + \frac{\sigma}{\rho}k^3\right)\tanh(kh)}$$

- σ: surface tension coefficient.
- ρ: water density.
- k: wave number.

Meaning

The k^3 term matters more for short waves. Surface tension is usually negligible for tsunamis, but important for ripples and laboratory waves.

Gravity or surface tension: a length scale

The capillary length compares surface tension with gravity:

$$\ell_c = \sqrt{\frac{\sigma}{\rho g}}$$

For water, this is only a few millimeters.

Modeling lesson

Before choosing an equation, ask: which physical effects are important at this scale?

Scale	Dominant effect	Typical model idea
millimeters	surface tension	capillary waves
meters	gravity + nonlinearity	coastal waves
kilometers	gravity + bathymetry	tsunamis

Nonlinearity: waves interact with themselves

A simple nonlinear equation is the inviscid Burgers equation:

$$\boxed{u_t + uu_x = 0}$$

- Large values of u move faster than small values.
- Profiles can steepen.
- Eventually shocks or breaking may appear in more physical models.

Coastal interpretation

Near shore, nonlinearity becomes more important because amplitude/depth becomes larger.

A bridge model: KdV idea

The Korteweg-de Vries equation balances nonlinearity and dispersion:

$$\boxed{\eta_t + c_0\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0}$$

$c_0\eta_x$	$\alpha\eta\eta_x$	$\beta\eta_{xxx}$
linear transport	nonlinear steepening	dispersive spreading

A solitary wave appears when steepening and dispersion balance.

Boussinesq-type thinking

Boussinesq models keep both nonlinearity and dispersion for shallow water waves.

$$\boxed{\text{shallow water} + \text{correction terms for dispersion}}$$

A schematic form is:

$$\eta_t + \nabla \cdot ((h+\eta)\mathbf{u}) = 0,$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g\nabla\eta + \text{dispersive terms} = 0.$$

Why useful?

They are more accurate than non-dispersive shallow-water equations when wavelength is long but not infinitely long.

Non-hydrostatic idea

Hydrostatic pressure assumes vertical acceleration is negligible.

$$\boxed{\text{pressure} \approx \text{weight of the water column}}$$

For some waves, vertical acceleration matters. Then pressure has a non-hydrostatic component.

$$p = p_{hydrostatic} + p_{nonhydrostatic}.$$

Engineering meaning

Non-hydrostatic models can capture dispersive behavior and more detailed wave motion without relying only on asymptotic formulas.

Computing idea 1: discretize space and time

Replace continuous variables by values on a grid:

$$\boxed{x_i = i\Delta x, \qquad t^n = n\Delta t, \qquad \eta_i^n \approx \eta(x_i, t^n)}$$

- Derivatives become differences.
- Equations become update rules.
- A computer repeats the update many times.

Numerical philosophy

A simulation is not magic. It is a large number of simple local calculations.

Computing idea 2: a derivative becomes a difference

For a function $q(x,t)$,

$$\boxed{q_x(x_i, t^n) \approx \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}}$$

For a time derivative,

$$\boxed{q_t(x_i, t^n) \approx \frac{q_i^{n+1} - q_i^n}{\Delta t}}$$

Student activity

Take a simple profile on five grid points and compute one approximate derivative by hand.

Computing idea 3: minimal simulation loop

```
set depth h(x)
set initial wave eta(x,0)
set initial velocity u(x,0)
choose dx and dt

for n = 0,1,2,...:
    compute spatial derivatives
    update eta using mass equation
    update u using momentum equation
    apply boundary conditions
    plot or store the result
```

mathematics $\rightarrow$ algorithm $\rightarrow$ code $\rightarrow$ physical interpretation

Stability: the CFL idea

A numerical method must respect the physical speed of information.

$$\Delta t \lesssim \frac{\Delta x}{\max c}$$

For shallow water,

$$c \approx \sqrt{gh}$$

Interpretation

During one time step, a wave should not travel too far across the grid. Otherwise the computation may become unstable.

Simple experiment: change the depth

Experiment A

Flat bottom:

$$h(x) = h_0.$$

Wave travels at almost constant speed.

Experiment B

Sloping bottom:

$$h(x) = h_0 - sx.$$

Wave slows down and may amplify near shore.

Same equation, different geology, different risk.

Simple experiment: add a source

A bottom deformation can be modeled as a source.

$$\boxed{b(x, t) = B(x)S(t)}$$

- $B(x)$: where the sea floor moves.
- $S(t)$: how fast it moves.
- Faster source usually creates a more energetic wave.

Discussion

What changes if the source is an earthquake, a landslide, or a volcanic collapse?

Simple experiment: surface tension on/off

Compare two dispersion relations:

$$\text{gravity only: } \omega^2 = gk \tanh(kh)$$

$$\text{gravity + surface tension: } \omega^2 = \left(gk + \frac{\sigma}{\rho}k^3\right) \tanh(kh)$$

Student question

For very small wavelengths, which term becomes dominant: gk or $(\sigma/\rho)k^3$?

From model to hazard: what do we output?

A computational model is useful when it produces interpretable quantities:

- arrival time,
- maximum water elevation,
- flow speed,
- inundation distance,
- sensitivity to bathymetry/source.

Risk language

$$\text{hazard} \neq \text{panic}$$

$$\text{hazard} = \text{information for decisions}$$

What makes a good mathematical model?

Good models are

- simple enough to understand,
- rich enough to describe reality,
- stable enough to compute,
- honest about uncertainty.

Bad models often

- include every effect without scale analysis,
- hide assumptions,
- confuse beautiful equations with useful predictions.

A small group activity

Task

Design a minimal model for a coastal wave problem.

1. Choose the phenomenon: storm wave, tsunami, ripple, or harbor wave.
2. Choose variables: η, u, h, source terms.
3. Choose important effects: geology, dispersion, surface tension, friction.
4. Decide: what output matters for safety or understanding?

The goal is not the perfect equation; the goal is a defensible modeling choice.

Where students can go next

Mathematics

PDEs, asymptotic analysis, conservation laws, numerical analysis, optimization.

Computation

Finite differences, finite volumes, finite elements, stability, visualization.

Applications

Coastal engineering, tsunami science, climate risk, marine energy, environmental modeling.

Research attitude

Ask simple questions deeply.

Take-home messages

1. Waves are a beautiful entrance to applied mathematics.
2. Tsunami speed begins with the simple formula $c = \sqrt{gh}$.
3. Geology enters through depth, bottom deformation, and coastal shape.
4. Surface tension matters at small scales; gravity dominates at large scales.
5. Computing is the translation of equations into stable update rules.

Mathematics helps us see, simulate, and prepare.

Thank you

Questions and discussion

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