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S.S Chern J.Moser Real hypersurfaces in complex manifolds, Acta Math. 133 (1974) 219 _ 271

On non degeneracy conditions for the Levi map in the hypersurface case. (1)

Contents:

Lecture 1; Real hypersurfaces in $\mathbb{C}^N$/vector fields.

$\mathbb{C}^N$ $Z = (Z_1, \ldots, Z_N)$ $Z_j = x_j + i y_j$, $j = 1, \ldots, N$

x_j, y_j real numbers $\overline{Z} = (\overline{Z}_1, \ldots, \overline{Z}_N)$, $\overline{Z}_j = x_j - i y_j$, complex conjugates of Z_j, $i^2 = -1$

$x = (x_1, \ldots, x_N)$ $y = (y_1, \ldots, y_N)$, $|Z_j|^2 = x_j^2 + y_j^2$

We may identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$, $\mathbb{C}^N \simeq \mathbb{R}^{2N}$

$$Z_1, \ldots, Z_N \longrightarrow x_1, x_2, \ldots, x_N, y_1, \ldots, y_N$$

$$\frac{Z_j + \overline{Z}_j}{2}, \quad \frac{Z_j - \overline{Z}_j}{2i}$$

So if we have a function on a subset of $\mathbb{C}^N$, we write

$$f(x, y) \text{ or } f(Z, \overline{Z}) \text{ !}$$

Comment: $N=1$, $\mathbb{C} \simeq \mathbb{R}^2$ $\quad y = f(x)$ curve

$\rightsquigarrow$ want to generalize this notion to $\mathbb{C}^N \simeq \mathbb{R}^{2N}$, using the complex structure

(No $\mathbb{R}^3$!)

Definition

A smooth real hypersurface in $\mathbb{C}^N$ is a subset M of $\mathbb{C}^N$ such that $\forall p_0 \in M$, $\exists\, \mathcal{U}$ neighborhood of p_0 in $\mathbb{C}^N$ and a smooth, real-valued function ρ defined in $\mathcal{U}$ such that

$$M \cap \mathcal{U} = \{ Z \in \mathcal{U} : \rho(Z, \bar{Z}) = 0 \},\ \text{with}$$

$$d\rho = \left(\frac{\partial \rho}{\partial x_1}(p), \ldots, \frac{\partial \rho}{\partial x_N}(p), \frac{\partial \rho}{\partial y_1}(p), \ldots, \frac{\partial \rho}{\partial y_N}(p) \right) \neq 0, \quad p \in \mathcal{U}$$

One say that M is real-analytic if ρ is real analytic.

Examples.

The unit sphere in $\mathbb{C}^N$ given by $\sum_1^N |z_j|^2 = 1$ is a compact hypersurface.

Exercice 1 Let $H(Z)$ be the holomorphic rational mapping given by

$$H_j(Z) = \frac{i\, Z_j}{1 - Z_N}, \quad j = 1, \ldots, N-1, \quad H_N(Z) = \frac{i(Z_N + 1)}{1 - Z_N}$$

Show that H sends the unit sphere minus the point $(0, 0, \ldots, 1)$ bijectively to the hypersurface

$$\operatorname{Im} Z_N = \sum_{j=1}^{N-1} |Z_j|^2$$

(called the Lewy hypersurface)

(Résolution page 5 / Thcm)

Exercice 2: For any hypersurface M and any $p_0 \in M$, there exist smooth coordinates $(x_1', \dots, x_{2N}')$ near $p_0 \in M$ such that M is given by

$$x_1' = 0 \quad \text{in a neighborhood of } p_0$$

(page 4, BER)

Exercice 3: If ρ and ρ' are two defining functions near $p_0 \in M$, there exists a non-vanishing real-valued smooth function a such that

$$\rho = a\rho' \quad \text{near } p_0$$

we have the following important result that shows that there exist "good" coordinates

Theorem: (without proof: can be found in BER p. 4).

Let M be a real hypersurface in $\mathbb{C}^{N}_{(n+1)}$ at p_0 then there are holomorphic coordinate (z, w) near p_0, $z \in \mathbb{C}^N$, $w \in \mathbb{C}$, $w = s + it$, $\varphi(z, \bar{z}, s)$ a real-valued smooth function such that $\varphi(z, \bar{z}, s)$ is defined near 0 in $\mathbb{R}^{2n+1}$,

$$\varphi(0) = 0, \quad d\varphi(0) = 0 \quad \text{and} \quad M \text{ is given near } p_0 = 0 \text{ by}$$

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)$$

or

$$t = \varphi(z, \bar{z}, s) \quad \leftarrow \left(\begin{array}{c}\text{curve} \\ \text{in } \mathbb{C}\end{array}\right)$$

no z!

The proof is based on the Taylor extension of the defining function.

Now, we are interested to use the complex structure of $\mathbb{C}^N$. (4

$$T_p \mathbb{C}^N = T_p \mathbb{R}^{2N} = \left\{ X = \sum_1^N a_j \frac{\partial}{\partial x_j}\Big|_p + \sum b_j \frac{\partial}{\partial y_j}\Big|_p, \; a_j, b_j \in \mathbb{R} \right\}$$

real tangent space to $\mathbb{C}^N$ at p

Definition If $p \in M$ and $X \in T_p\mathbb{C}^N$, we say that X is tangent to M at p if

$$\sum_{j=1}^N a_j \frac{\partial \rho}{\partial x_j}(p,\bar{p}) + \sum_{j=1}^N b_j \frac{\partial \rho}{\partial y_j}(p,\bar{p}) = 0 \leftarrow \mathbb{R}^{2N} \text{ language}$$

Now $\mathbb{C}^N \simeq \mathbb{R}^{2N}$. Define the "complexified" tangent space $\mathbb{C}T_p\mathbb{C}^N$ et $\mathbb{C}T_pM$ by allowing the coefficients a_j and b_j in the expression above to be complex numbers.

Remarks:

$$\dim_{\mathbb{R}} T_pM = \dim_{\mathbb{C}} \mathbb{C}T_pM = 2N-1$$

As usual
$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Therefore $X \in \mathbb{C}T_p\mathbb{C}^N$ can be written uniquely in the form

$$X = \sum a_j \frac{\partial}{\partial z_j}\Big|_p + \sum b_j \frac{\partial}{\partial \bar{z}_j}\Big|_p \quad , a_j, b_j \in \mathbb{C}$$

X is said holomorphic if $b_j = 0$ $T_p^{1,0}\mathbb{C}^N$
antiholomorphic if $a_j = 0$ $T_p^{0,1}\mathbb{C}^N$
$\dim_{\mathbb{C}} T_p^{1,0}\mathbb{C}^N = \dim_{\mathbb{C}} T_p^{0,1}\mathbb{C}^N = N$

Definition

$$\mathcal{V}_p = T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM$$

Exercice 4:

$$\dim_{\mathbb{C}} \mathcal{V}_p = N-1$$

(reason $d\rho = \partial\rho + \bar{\partial}\rho$

$$\sum \frac{\partial\rho}{\partial x_j} dx_j + \sum \frac{\partial\rho}{\partial y_j} dy_j = \sum \frac{\partial\rho}{\partial z_j} dz_j + \sum \frac{\partial\rho}{\partial \bar{z}_j} d\bar{z}_j$$

[Reason: $\frac{1}{2}\left(\frac{\partial\rho}{\partial x} - i\frac{\partial\rho}{\partial y}\right)(dx + i\,dy) + \frac{1}{2}\left(\frac{\partial\rho}{\partial x} + i\frac{\partial\rho}{\partial y}\right)(dx - i\,dy)$

$= \frac{\partial\rho}{\partial x}dx + \frac{\partial\rho}{\partial y}dy$!]

Definition Let M be a real hypersurface and let $\mathcal{V}$ be its CR bundle,

‖($\mathcal{V} \subset \mathbb{C}TM$ whose fiber at $p \in M$ is $\mathcal{V}_p$ is called the CR bundle of M)

A vector field L on M is a CR vector field if for every $p \in M$, $L_p \in \mathcal{V}_p$

$$L = \sum_1^N c_j(p) \frac{\partial}{\partial \bar z_j}, \quad c_j \text{ smooth functions on } M.$$

and $\sum c_j(p) \frac{\partial \rho}{\partial \bar z_j}(p) = 0$.

Definition: Let M be a hypersurface

f is a CR function if $Lf = 0$, $\forall$ CR vector field L on M, $f \in C^1(M)$ at least.

Basis for the vector fields:

$$L_j = \frac{\partial}{\partial \bar z_j} - \frac{\rho_{\bar z}}{\rho_{\bar w}} \frac{\partial}{\partial \bar w}, \quad j = 1, \dots, n$$

Indeed, $\left(\frac{\partial}{\partial \bar z_j} - \frac{\rho_{\bar z_j}}{\rho_{\bar w}} \frac{\partial}{\partial \bar w}\right)\rho = \frac{\partial \rho}{\partial \bar z_j} - \rho_{\bar z_j} = 0!$

We have that if M is given by $\operatorname{Im} w = \varphi(z, \bar z, \operatorname{Re} w)$, $\varphi(0) = 0$, $d\varphi(0)$,

$$L_j = \frac{\partial}{\partial \bar z_j} - 2i \frac{\varphi_{\bar z_j}}{1 + i\varphi_s} \frac{\partial}{\partial \bar w}, \quad s = \operatorname{Re} w$$

$$\begin{cases} \left(\frac{\partial}{\partial \bar z_j} - 2i \frac{\varphi_{\bar z_j}}{1 + i\varphi_s} \frac{\partial}{\partial \bar w}\right)\left(\frac{w - \bar w}{2i} - \varphi(z, \bar z, \operatorname{Re} w)\right)^{\frac{w+\bar w}{2}} \\ = -\varphi_{\bar z_j} - 2i \frac{\varphi_{\bar z_j}}{1 + i\varphi_s}\left(\frac{-1}{2i} - \varphi_s\left(\frac{1}{2}\right)\right) \end{cases}$$

$$= -\varphi_{\bar{z}_j} - 2i\,\frac{\varphi_{\bar{z}_j}}{1+i\varphi_s}\left(-\frac{1}{2i} - \varphi_s\,\frac{i}{2i}\right)$$

$$= -\varphi_{\bar{z}_j} + \frac{\varphi_{\bar{z}_j}}{\cancel{1+i\varphi_s}}\left(\cancel{1+i\varphi_s}\right) = 0$$

<u>Exercice 5:</u> (Very important)

In the parametrization $\psi : (z, \bar{z}, s) \to (z, s+i\varphi(z,\bar{z},s))$ the pushforward by ψ^{-1} is given by

$$\Lambda_j = \frac{\partial}{\partial \bar{z}_j} - i\,\underbrace{\frac{\varphi_{\bar{z}_j}}{1+i\varphi_s}\,\frac{\partial}{\partial s}}_{e}, \quad j = 1, \dots, n$$

<u>Proof:</u> $(\psi^*)^{-1}(L_j)(f) = L_j(f \circ \psi^{-1})$

$j=1$

$$\left(\frac{\partial}{\partial \bar{z}} - 2i\,\frac{\varphi_{\bar{z}}}{1+i\varphi_s}\right)\frac{\partial}{\partial \bar{w}}\left(\tilde{f}(\psi^{-1}(z, \bar{z}, s+i\varphi)\right)$$

or directly

$$\psi^*\left(\frac{\partial}{\partial \bar{z}} - i\,\frac{\varphi_{\bar{z}}}{1+i\varphi_s}\,\frac{\partial}{\partial s}\right) f\,(z, \bar{z}, s+i\varphi(z,\bar{z},s))$$

$$= \left(\frac{\partial}{\partial \bar{z}} - i\,\frac{\varphi_{\bar{z}}}{1+i\varphi_s}\,\frac{\partial}{\partial s}\right)\tilde{f}\,(z, \bar{z}, s)$$

$$= \frac{\partial}{\partial \bar{z}} f(z, \bar{z}, s+i\varphi(z,\bar{z},s)) + \frac{\partial}{\partial w} f\,(i\varphi_{\bar{z}})$$

$$+ \frac{\partial}{\partial \bar{w}} f\,(-i\varphi_{\bar{z}})$$

$$- i \frac{\varphi_{\bar z}}{1+i\varphi_s} \left[\frac{\partial}{\partial w} f(\ )(1+i\varphi_s) + \frac{\partial}{\partial \bar w} f \ (1-i\varphi_s) \right]$$

$$= \frac{\partial}{\partial \bar z} f + \frac{\partial}{\partial w} f \ (i\varphi_{\bar z}) + \frac{\partial}{\partial \bar w} f \ (-i\varphi_{\bar z}) \frac{1+i\varphi_s}{1+i\varphi_s}$$

$$- i \ \varphi_{\bar z} \frac{\partial f}{\partial w} - i \ \varphi_{\bar z} \ f \ (1-i\varphi_s) / _{1+i\varphi_s} \frac{\partial f}{\partial \bar w}$$

$$= \frac{\partial f}{\partial \bar z} - \frac{i\varphi_{\bar z}}{1+i\varphi_s} \left(\underbrace{1+i\varphi_s + 1 - i\varphi_s}_{2} \right) \frac{\partial f}{\partial \bar w}$$

$$= \left(\frac{\partial}{\partial \bar z} - \frac{2i\,\varphi_{\bar z}}{1+i\varphi_s} \frac{\partial}{\partial \bar w} \right) f \quad \checkmark$$

Lecture IV:

Levi form of a real hypersurface in $\mathbb{C}^N$.

$\boxed{N = n+1}$

Rappel: $\mathcal{V}_p = T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM$

$$L_j = \frac{\partial}{\partial \bar{z}_j} - 2i\frac{\varphi_{\bar{z}_j}}{1+i\varphi_s}\frac{\partial}{\partial \bar{w}}$$

$$\Lambda_j = \frac{\partial}{\partial \bar{z}_j} - i\frac{\varphi_{\bar{z}_j}}{1+i\varphi_s}\frac{\partial}{\partial s}$$

Let π_p be the natural quotient map

$$\pi_p : \mathbb{C}T_pM \longrightarrow \mathbb{C}T_pM / \mathcal{V}_p \oplus \overline{\mathcal{V}_p}$$

and

$$\pi : \mathbb{C}TM \longrightarrow \mathbb{C}TM / \mathcal{V} \oplus \overline{\mathcal{V}}$$

Exercice 6: If X and Y are CR vector fields on M, defined near p, with $X(p) = X_p \in \mathcal{V}_p$ and $Y(p) = Y_p \in \mathcal{V}_p \Longrightarrow$

$\pi_p\left([X, \bar{Y}](p)\right)$ depends only on the values X_p and Y_p and not of the choice of the vector fields X and Y extending X_p and Y_p

<u>Proof:</u>

$$\left[a(z,\bar z)\left(\frac{\partial}{\partial \bar z_j} - i\,\frac{\varphi_{\bar z_j}}{1+i\varphi_s}\,\frac{\partial}{\partial s}\right),\; b(z,\bar z)\left(\frac{\partial}{\partial z_k} + i\,\frac{\varphi_{z_k}}{1-i\varphi_s}\,\frac{\partial}{\partial s}\right)\right]$$

$$= a(z,\bar z)\, L(b)\left[\frac{\partial}{\partial z_k} + i\,\frac{\varphi_{z_k}}{1-i\varphi_s}\,\frac{\partial}{\partial s}\right] \qquad \text{mod}$$

$$- \; b(z,\bar z)\, \bar L(a)\left[\frac{\partial}{\partial \bar z_j} - i\,\frac{\varphi_{\bar z_j}}{1+i\varphi_s}\,\frac{\partial}{\partial s}\right]$$

$$+ \; a(z,\bar z)\, b(z,\bar z)\left(\text{terms in } \varphi \text{ and } \frac{\partial}{\partial s}\right)$$

Hence the following definition

<u>Definition</u> The Levi form at $p \in M$ (called also Levi map in more general context) is the Hermitian form

$$\mathcal{L}_p : V_p \times V_p \longrightarrow \mathbb{C}T_pM \,/\, V_p \oplus \overline{V_p}$$

$$(X_p, Y_p) \longrightarrow \frac{1}{2i}\,\pi_p\left[X, \bar Y\right](p)$$

where X and Y are CR vector fields on M extending the CR vectors X_p and Y_p.

$\mathcal{L} : V \times V \to \mathbb{C}TM / V \oplus \overline{V}$

Levi map

Exercise 7 (part of the claim) Show that $\mathcal{L}_p$ is an Hermitian form on V_p.

We have to show:

- $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$
- $\langle y, x \rangle = \overline{\langle x, y \rangle}$

$$\langle a X_p + b \tilde{X}_p , Y_p \rangle = \frac{1}{2i} \pi_p \left([a X_p + b \tilde{X}_p , \overline{Y}_p] \right)$$

$= \dots$ bracket properties.

- $$\langle Y_p, X_p \rangle = \frac{1}{2i} [Y_p, \overline{X}_p]$$

$$\langle X_p, Y_p \rangle = \frac{1}{2i} [X_p, \overline{Y}_p]$$

$$\Rightarrow \overline{\langle X_p, Y_p \rangle} = -\frac{1}{2i} [\overline{X}_p, Y_p] = \frac{1}{2i} [Y_p, \overline{X}_p]$$

$= \langle Y_p, X_p \rangle$. ✓

(That is why one needs to put a "i" in front of the bracket)

(4

$d\varphi = 0$
$\varphi(0) = 0$

Exercice 8:

Let M in $\mathbb{C}^2$ be given by

$$\operatorname{Im} w = 2 z\bar{z} + z^2\bar{z}^2 + z^3\bar{z}^3$$

Compute the Levi form of M at 0.

Proof:

in $\mathbb{C}^2$:
$$L = \frac{\partial}{\partial \bar{z}} - \frac{i\,\varphi_{\bar{z}}}{1+i\,\varphi_s}\frac{\partial}{\partial s}$$
$$\bar{L} = \frac{\partial}{\partial z} + i\,\frac{\varphi_z}{1-i\,\varphi_s}\frac{\partial}{\partial s}$$

Take a basis L: we compute $[L, \bar{L}]$

$$\frac{1}{2i}[L,\bar{L}] = \left[\frac{\partial}{\partial \bar{z}} - i\frac{\varphi_{\bar{z}}}{1+i\varphi_s}\frac{\partial}{\partial s}, \frac{\partial}{\partial z} + i\frac{\varphi_z}{1-i\varphi_s}\frac{\partial}{\partial s}\right]$$

$$= \left[\frac{\partial}{\partial \bar{z}} - i\,\frac{2z + 2z^2\bar{z} + 3z^3\bar{z}^2}{1}\frac{\partial}{\partial s}, \frac{\partial}{\partial z} + i\,\frac{2\bar{z} + 2\bar{z}^2 z + 3\bar{z}^3 z^2}{1}\frac{\partial}{\partial s}\right]$$

$$= i\left(2 + 4\bar{z}^2 z + 9\bar{z}^2 z^2\right)\frac{\partial}{\partial s} + i\left(2 + 4z^2\bar{z} + 9z^2\bar{z}^2\right)\Big|_0$$

$$= \frac{1}{2i} 4i = 2$$

$\rightsquigarrow$ $\operatorname{Im} w = a z\bar{z} + \dots$

a

In general in $\mathbb{C}^{n+1}$

$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) = a z_1 \bar{z}_1 + b z_2 \bar{z}_2 + c z_1 \bar{z}_2 + d z_2 \bar{z}_1 + \ldots$

$\begin{pmatrix} a & c \\ d & b \end{pmatrix}$ is the matrix of the Hermitian form.

Exercise 8: Let M in $\mathbb{C}^3$ given by

$$\operatorname{Im} w = z_1 \bar{z}_1 + s z_1 + s \bar{z}_1 + z_2 \bar{z}_2$$

Compute the Levi form of M at 0

Definition. The Levi form $\mathcal{L}_p$ is nondegenerate if

$$\mathcal{L}_p(X_p, Y_p) = 0 \quad \forall Y_p \in \mathcal{V}_p \Rightarrow X_p = 0$$

Examples $\operatorname{Im} w = z^2 \bar{z}^2$ in $\mathbb{C}^2$ is degenerate!

$\operatorname{Im} w = z_1 \bar{z}_1$ in $\mathbb{C}^3$ is degenerate

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{L}_p(\underbrace{L_2}, L) = 0! \quad \forall L$$

$$\left[\frac{\partial}{\partial \bar{z}_2}, \frac{\partial}{\partial z_2} \right] = 0$$

$$\left[\frac{\partial}{\partial \bar{z}_2}, \frac{\partial}{\partial z_1} + i \bar{z}_1 \frac{\partial}{\partial s} \right] = 0 \,!$$

Definition. M is pseudoconvex at 0 if the Levi form is either positive semidefinite at all p in a neighborhood of 0 in M or negative semidefinite at all p in such a neighborhood.

Similarly, M is strictly pseudoconvex at $p_0 \in M$ if the Levi form is positive or negative definite at $p_0 \in M$

Examples. In $\mathbb{C}^2$, Levi nondegenerate hypersurfa are pseudoconvex ; $a z\bar{z}$

$a > 0$ < 0 or $= 0$

In $\mathbb{C}^3$

$\operatorname{Im} w = |z_1|^2 + |z_2|^2$ strictly pseudo convex

$\operatorname{Im} w = |z_1|^2$ pseudo convex

$\operatorname{Im} w = |z_1|^2 - |z_2|^2$ Levi nondegenerate!

$\begin{pmatrix} & \\ & \end{pmatrix}$

One has Levi nondg $\iff \det(\ ) \neq 0$

Appendix: in case enough time about CR functions

Let M be a real-analytic hypersurface of $\mathbb{C}^N$.
If $u(z,\bar z,s)$ is a CR function, which means

$$L_j u = 0 \quad \forall j = 1,\dots,n \quad , N = n+1,$$

then $u(z,\bar z,s)$ extends holomorphically to a full neighborhood of 0 if and only if there exists $\varepsilon > 0$ such that for every $z \in \mathbb{C}^n$, $|z| < \varepsilon$, the function $s \to u(z,\bar z,s)$ extends holomorphically to the open set $\{s+it \in \mathbb{C}, |s| < \varepsilon, |t| < \varepsilon\}$ in such a way that the extension $u(z,\bar z,s+it)$ is a bounded, mesurable function in all its variables

Proof:

As M is real-analytic, $\varphi(z,\bar z,s)$ is real-analytic, and hence s can be complexified.
Therefore the map

$$\Psi \quad (z,\bar z,s) \longrightarrow (z, s+i\varphi(z,\bar z,s))$$

$$\Downarrow$$

$$\Psi \quad (z,\bar z,s,t) \to (z, s+it+i\varphi(z,\bar z,s+it))$$

If f is a CR function on M, then

$$u(z,\bar z,s) = f(z, s+i\varphi(z,\bar z,s)) = f(\Psi(z,\bar z,s))$$

is "a CR" function

Def: $u(z,\bar z,s)$ extends holomorphically to a neighborhood of 0 in $\mathbb{C}^N$ if f does

i.e. $\exists\, h(z,w)$ holomorphic in a neighborh. of $0 \in \mathbb{C}^N$ such that

$$u(z,\bar z,s) = h(z, s+i\varphi(z,\bar z,s))$$

$\Rightarrow$ $\exists$ a holomorphic function $h(z,w)$ in a neighborhood of 0 in $\mathbb{C}^{n+1}$ such that $h(z, s+i\varphi(z,\bar{z},s)) = u(z,\bar{z},s)$

Hence, if u extends holomorphically to a neighborhood of 0, then $u(z,\bar{z},s)$ extends holomorphically in s for fixed z by putting

$$u(z,\bar{z},s+it) = h(z, s+it+i\varphi(z,\bar{z},s+it)) = h(\psi(x,y,s,t))$$

in this structure.

$\Leftarrow$ $s \to u(z,\bar{z},s)$ extends holomorphically for fixed z, $|z|<\varepsilon$

we define $h(z,\bar{z},w,\bar{w})$ by

$$h(z,\bar{z},w,\bar{w}) = u(\psi^{-1}(z,w))$$

By hypothesis, h is measurable, bounded function.

- This is a very nice argument of distribution theory.

BER $\to$ p. 28!

<u>Corollary</u> (Important for the third lecture)

Let f be a CR function

then f extends as a holomorphic function if and only if f is real analytic in a neigh. of p in M.

Infinitesimal CR Automorphisms Lecture III

Notion of smooth vector field in the first lecture.

Definition: A smooth real vector field X defined in an open subset U of M is called infinitesimal CR automorphism of M if, for every $p \in U$, there exists $\varepsilon > 0$ and an open neighborhood $U' \subset U$ of p such that the diffeomorphisms $U' \ni q \to \exp tX \cdot q \in U$ are CR mappings for each $|t| \leq \varepsilon$.

[The flow $t \to \exp tX \cdot p$ is the integral curve $\gamma(t)$ satisfying the initial value problem

$$\begin{cases} \dfrac{d\gamma}{dt}(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases}$$

Here is a very useful characterization of the infinitesimal CR automorphisms of M (without proof BER p 362)

Theorem.

Let X be a real vector field defined in an open subset $U \subset M$

then X is an infinitesimal CR automorphism of M $\iff$

for every smooth CR vector field L in U, the commutator $[X, L]$ is again a CR vector field.

Definition $\mathrm{aut}(M, p)$ is the set of germs of infinitesimal CR automorphisms at p

A consequence of the theorem above is that $\mathrm{aut}(M, p_0)$ is a Lie algebra.

Indeed if $X, Y \in \mathrm{aut}(M, p_0)$, then $[X, Y] \in \mathrm{aut}(M, p_0)$

$X \in \mathrm{aut}(M, p) \Rightarrow [X, L]$ is CR if L is CR

$Y \in \mathrm{aut}(M, p) \Rightarrow [Y, L]$ is CR " ...

$$\big[[X, Y], L \big] = - \big[L, [X, Y] \big] = \underbrace{\big[Y, [L, X] \big]}_{CR} + \underbrace{\big[X, [Z, L] \big]}_{CR}$$

Corollary: Let M be a smooth real hypersurface (3
and let X be a smooth real vector field on M defined in an open subset $U \subset M$. Then X is an infinitesimal CR automorphism of M if and only if

$$X = \sum_{j=1}^{N} \left(a_j \frac{\partial}{\partial z_j} + \bar{a}_j \frac{\partial}{\partial \bar{z}_j} \right),$$

where the a_j, $j = 1, \dots, N$ are CR functions in M.

Proof: Observe that X, being a real smooth vector field on $M \subset \mathbb{C}^N$ has a representation of the form

$$X = \sum_{j=1}^{N} \left(a_j \frac{\partial}{\partial z_j} + \bar{a}_j \frac{\partial}{\partial \bar{z}_j} \right)$$

for some smooth functions a_j in M. We must show that X is an infinitesimal CR automorphism if and only if the a_j, $j = 1, \dots, N$ are CR functions

$$L = \sum_{j=1}^{N} b_j \frac{\partial}{\partial \bar{z}_j} \quad \text{CR vector field}$$

$\Rightarrow$ theorem above

$$[X, L] = \sum_{j=1}^{N} (X b_j - L \bar{a}_j) \frac{\partial}{\partial \bar{z}_j} - L a_j \frac{\partial}{\partial z_j}$$

Indeed

$$\left[\sum a_j \frac{\partial}{\partial z_j} + \sum \bar{a}_j \frac{\partial}{\partial \bar{z}_j} \,,\ \sum b_j \frac{\partial}{\partial \bar{z}_j} \right] =$$

$$\sum \left[X(b_j) - L\bar{a}_j \right] \frac{\partial}{\partial \bar{z}_j} - \underbrace{L a_j}_{=0} \frac{\partial}{\partial z_j}$$

$$\Leftrightarrow L a_j = 0 \Longleftrightarrow a_j \text{ CR}$$

#

Now, we will be particularly interested by the following Lie subalgebra of $aut(M,p)$

Definition: $hol(M,p) \subset aut(M,p)$ is the set of germ of real analytic infinitesimal CR automorphism

M real-analytic

Proposition. $X \in hol(M,p)$ (M real-analytic) $\Leftrightarrow$ there exists Z holomorphic vector field such that $\mathrm{Re}\, Z$ is tangent to M and $X = \mathrm{Re}\, Z$.

Proof: By the corollary proved in Lecture II, we have that $X \in hol(M,p)$ is given by

$$\sum h_j(z) \frac{\partial}{\partial z_j} + \sum \overline{h_j(z)} \frac{\partial}{\partial \bar{z}_j} \Bigg|_M$$

$$a_j = h_j(z, \bar{z}, s + i\varphi(z,\bar{z},s))$$

It shows also that $hol(M,p)$ is a Lie subalgebra: $X, Y \in hol(M,p) \Longrightarrow [X,Y] \in hol(M,p)$

Indeed: $\left[\sum \underbrace{h_j}_{hol} \frac{\partial}{\partial z_j} + \sum \overline{h_j} \frac{\partial}{\partial \overline{z_j}}, \sum k_j \frac{\partial}{\partial z_j} + \sum \overline{k_j} \frac{\partial}{\partial \overline{z_j}}\right]$

$= \left[\sum h_j \frac{\partial}{\partial z_j}, \sum k_j \frac{\partial}{\partial z_j}\right] + \left[\sum \overline{h_j} \frac{\partial}{\partial \overline{z_j}}, \sum \overline{k_j} \frac{\partial}{\partial \overline{z_j}}\right]$

$= \mathrm{Re}\left[\sum h_j \frac{\partial}{\partial z_j}, \sum k_j \frac{\partial}{\partial z_j}\right]$ hol

In the next lecture, we will compute $hol(M,p)$ for M given by

$\mathrm{Im}\, w = a z\bar{z} \qquad (z,w) \in \mathbb{C}^2$

$a \neq 0$ Levi non degenerate

Lecture IV

Computing hol$(M,0)$ for M given by

$$\operatorname{Im} w = c z\bar{z} \qquad c \neq 0 \qquad (z,w) \in \mathbb{C}^2$$

For simplicity, I am choosing the $\mathbb{C}^2$ case, but the computation is the same in $\mathbb{C}^N$

$$M : \operatorname{Im} w = z\bar{z}$$

- To make the computation easier, we decide to give weights to the variables z, w and the derivatives $\frac{\partial}{\partial z}, \frac{\partial}{\partial w}$

$$z \rightsquigarrow \frac{1}{2} \qquad w \rightsquigarrow 1 \qquad \frac{\partial}{\partial z} \rightsquigarrow -\frac{1}{2}, \ \frac{\partial}{\partial w} \rightsquigarrow (-1)$$

$$\left(\text{or } z \rightsquigarrow 1 \qquad w \rightsquigarrow 2 \qquad \frac{\partial}{\partial z} \rightsquigarrow -1, \ \frac{\partial}{\partial w} \rightsquigarrow -2 \right)$$

it depends of the authors.

So, you see

$z\bar{z} \rightsquigarrow 1$ which is the weight of w!

So we care for holomorphic vector field $Z = \underbrace{f(z,w)}_{hol} \frac{\partial}{\partial z} + \underbrace{g(z,w)}_{hol} \frac{\partial}{\partial w}$

for which $\operatorname{Re} Z = X$ is tangent to M!

$$\operatorname{Re} Z\,(\operatorname{Im} w - z\bar{z}) = 0$$

$$Z(\operatorname{Im} w - z\bar{z}) + \bar{Z}(\operatorname{Im} w - z\bar{z}) = 0 \ !$$

$z(\ ) + \overline{z(\ )} = z(\ ) + \bar{z}(\ )$ 3

because it's real

What is the weight minimum of a vector field with these choice of weights?

$$\underbrace{a\frac{\partial}{\partial w}}_{-1} \quad ?$$

$\operatorname{Re} a\frac{\partial}{\partial w}(\operatorname{Im} w - z\bar{z}) = 0$ for which a?

$$a\frac{\partial}{\partial w}\left(\frac{w-\bar{w}}{2i} - z\bar{z}\right) = a\frac{1}{2i}$$

$$a\frac{1}{2i} + \bar{a}\frac{1}{-2i} = 0 \qquad a - \bar{a} = 0 \Rightarrow a = \bar{a} \Rightarrow$$

$a \in \mathbb{R}$

- $a\frac{\partial}{\partial w} \in \operatorname{hol}(M,0)$, $a \in \mathbb{R}$ 1

Next:

$$\underbrace{a\frac{\partial}{\partial z}}_{-\frac{1}{2}} + \underbrace{bz\frac{\partial}{\partial w}}_{-\frac{1}{2}} (\operatorname{Im} w - z\bar{z}) \qquad -1$$

$$= \left(a\frac{\partial}{\partial z} + bz\frac{\partial}{\partial w}\right)\left(\frac{w-\bar{w}}{2i} - z\bar{z}\right)$$

$$= -a\bar{z} + bz\frac{1}{2i}$$

$$\Rightarrow -a\bar{z} + bz\frac{1}{2i} - \bar{a}z + \bar{b}\bar{z}\left(-\frac{1}{2i}\right) = 0$$

$$-a\bar{z} + bz\frac{1}{2i} - \bar{a}z + \bar{b}\bar{z}\left(-\frac{1}{2i}\right) = 0 \quad |2i$$ ③

$$-\underbrace{2ia\bar{z}}_{} + bz - 2i\bar{a}z - \bar{b}\bar{z} = 0$$

$$-2ia - \bar{b} = 0 \qquad 2i\bar{a} - b = 0$$

$$b - 2i\bar{a} = 0 \qquad b - 2i\bar{a} = 0$$

hence, does not bring anything

$$b = 2i\bar{a} \quad , b, a \in \mathbb{C} \,!$$

$$\Rightarrow \bullet \; X = a\frac{\partial}{\partial z} + 2i\bar{a}z\frac{\partial}{\partial w} \quad , a \in \mathbb{C}$$

2 parameters

Check: ②

$$-\underbrace{a\bar{z}}_{} + \underbrace{\cancel{2i}\bar{a}z}_{}\frac{1}{\cancel{2i}} - \underbrace{\bar{a}z}_{} - \underbrace{\cancel{2i}a\bar{z}}_{}\left(-\frac{1}{\cancel{2i}}\right) =$$

$$a = 1 \qquad b = i$$

$$(a + ib)\frac{\partial}{\partial z} + 2i(a - ib)\frac{\partial}{\partial w}$$

$$b = 0 \qquad a\frac{\partial}{\partial z} + 2ia\frac{\partial}{\partial w}$$

$$a = 0 \qquad ib\frac{\partial}{\partial z} + 2i(-ib)\frac{\partial}{\partial w} = ib\frac{\partial}{\partial z} + 2b\frac{\partial}{\partial w}$$

<u>Basis:</u> $\frac{\partial}{\partial z} + 2i\frac{\partial}{\partial w}\,; \quad i\frac{\partial}{\partial z} + 2\frac{\partial}{\partial w}$

$$\dim_{\mathbb{R}} = 2$$

$-\frac{1}{2}$

Next weight 0

$$a z \frac{\partial}{\partial z} + (b z^2 + c w) \frac{\partial}{\partial w}$$

and you play the same game

$$-a \underset{\sim}{z} \bar{z} + (b \underset{\sim}{z}^2 + c w) \frac{1}{2i} - \bar{a} \underset{\sim}{z \bar{z}} + (\bar{b} \bar{z}^2 + \bar{c} \bar{w}) \left(-\frac{1}{2i}\right) \Big|_{M} = 0$$

$$(-a - \bar{a}) z \bar{z} + b z^2 \frac{1}{2i} - \bar{b} \bar{z}^2 \frac{1}{2i}$$

$$+ c w \frac{1}{2i} - \bar{c} \bar{w} \frac{1}{2i} \Big|_{M} = 0$$

that means that we have to replace w by $s + i z \bar{z}$

$$\Rightarrow (-a - \bar{a}) \underset{\sim}{z \bar{z}} + b z^2 \left(\frac{1}{2i}\right) - \bar{b} \bar{z}^2 \frac{1}{2i} + c(s + i \underset{\sim}{z \bar{z}}) \frac{1}{2i}$$

$$- \bar{c} (s - i \underset{\sim}{z \bar{z}}) \frac{1}{2i} = 0$$

$$-a - \bar{a} + \frac{1}{2} c + \frac{1}{2} \bar{c} = 0$$

$$b = 0$$

$$\frac{c}{2i} - \frac{\bar{c}}{2i} = 0 \Rightarrow c - \bar{c} = 0 \Rightarrow c \in \mathbb{R}$$

$$2(a + \bar{a}) = 2c \Rightarrow a + \bar{a} = c \qquad 2 \operatorname{Re} a = c$$

$$\underline{a = \frac{c}{2} + i d \quad d \in \mathbb{R}}$$

$$\Rightarrow \quad X = \left(\frac{c}{2} + i d\right) z \frac{\partial}{\partial z} + c w \frac{\partial}{\partial w}$$

$$d = 0 \qquad X = c \underbrace{\left(\frac{1}{2} z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}\right)}_{\text{Euler field or grading element}}$$

$c = 0 \qquad i\,d\, z\frac{\partial}{\partial z}$

$\Rightarrow$ Euler fiel + purely imaginary field

Now, we start we a crucial lemma

We have seen that $X = \frac{\partial}{\partial w} \in hol(M,0)$
we also know that $hol(M,0)$ is a Lie subalgebra, that is, if $X, Y \in hol(M,0)$ then $[X,Y] \in hol(M,0)$

Lemma: Let $X = a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w} \in hol(M,0)$ ($a(z,w)$ and $b(z,w)$ are holomorphic) of weight $\mu \geq 0$ otherwise there is no w!

$a(z,w) = \alpha_k(z) w^k + a_{k-1}(z) w^{k-1} + \dots a_0(z)$

$b(z,w) = b_e(z) w^e + b_{e-1}(z) w^{e-1} + \dots + b_0(z)$

$k > e \Rightarrow \quad Y = k! a_k(z)\frac{\partial}{\partial z} \in hol(M,0)$

$k = e \qquad Y = k! a_k(z)\frac{\partial}{\partial z} + e! b_e \frac{\partial}{\partial w} \in hol(M,0)$

$k < e \qquad Y = e! b_e(z)\frac{\partial}{\partial w} \in hol(M,0)$

Comment: You just differentiate the coeffici until you have no w!

Proof:

Take $X = a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w} \in hol(\Gamma,0)$

and $\frac{\partial}{\partial w} \in hol(\Gamma,0)$

$$\Rightarrow \left[\frac{\partial}{\partial w}, X\right] = \left[\frac{\partial}{\partial w}, a(z,w)\frac{\partial}{\partial z} + b(z,w)\frac{\partial}{\partial w}\right]$$

$$= \frac{\partial}{\partial w}a(z,w)\frac{\partial}{\partial z} + \frac{\partial}{\partial w}b(z,w) \in hol(\Gamma,0)$$

... and we may continue the process until it stops! #

Example: $X = \frac{1}{2}z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}$

$\Rightarrow \frac{\partial}{\partial w} \in hol(\Gamma,0)$: just differentiate!

Consequence It is enough to consider the vector fields whose coefficients do not contain any w! And then integrate them.

It then remains to understand the so called rigid vector fields of weight ≥ 0. Do they exist or do we have all with

$$\frac{\partial}{\partial w}, \quad a\frac{\partial}{\partial z} + 2i\bar{a}z\frac{\partial}{\partial w} \quad idz\frac{\partial}{\partial z}$$

$$1 \qquad\qquad 2 \qquad\qquad 1$$

$$a z^k \frac{\partial}{\partial z} + b z^u \frac{\partial}{\partial w}$$

$(k-1)/2 = u/2 - 1 = \mu$ weight of the vector field, $\mu > 0$

$\Rightarrow k > 1$

$$-a z^k \bar{z} + \underbrace{b z^u}_{=0} \frac{1}{2i} - \bar{a} \bar{z}^k z + \underbrace{\bar{b} \bar{z}^u}_{=0} \left(-\frac{1}{2i}\right) = 0$$

$\Rightarrow a = 0$ since $k > 1$

There are no and we have everything.

Exercise: Show that there is 1 integration of $a \frac{\partial}{\partial z} + 2i \bar{a} z \frac{\partial}{\partial w}$ but not more and 2 integration of $\frac{\partial}{\partial w}$ and not more.

Show that the 2-integration is given by $a\left[w^2 \frac{\partial}{\partial w} + zw \frac{\partial}{\partial z}\right]$, $a \in \mathbb{R}$ $\dim = 1$

Conclusion: the coefficient of all those holomorphic vector fields are at most 2

Very important for the next and final lecture!

Final Lecture: Lecture V

Chern - Moser Theory for Levi non degenerate hypersurfaces.

In the previous lecture, we have computed in detail $\mathrm{hol}(M,0)$ for

$$M \text{ given by } \mathrm{Im}\, w = z\bar{z} \;, (z,w) \in \mathbb{C}^2$$

But why? why is it important that the coefficients are at most of degree 2?

Here is the problem we want to solve

Let: M a real hypersurface Levi non degenerate at 0 in $\mathbb{C}^2$ given by

$$M: \;\; \mathrm{Im}\, w = \varphi(z, \bar{z}, \mathrm{Re}\, w) \;, \; \varphi(0) = 0,$$

$d\varphi(0) = 0$: no terms in $z, \bar{z}, s$!

In fact $\mathrm{Im}\, w = z\bar{z}$ + higher order terms

Remember that we put weights for z and $w = s+it$

$$z \sim \frac{1}{2} \qquad w, s, t \sim 1$$

Therefore

weight 1 $\qquad$ weight > 1 !

We are interested to biholomorphic map in a neighborhood of 0 in $\mathbb{C}^N$, $N = n+1$, sending M to itself

$$H: \quad M \to M$$

What are the properties of those maps?

For a preview, let us recall the following theorem of Cartan.

Theorem (Cartan 1931)

Let $\Omega \subset \mathbb{C}^n$ be a bounded connected open set containing 0 and

$F: \Omega \to \Omega$ be a holomorphic map satisfying

- $F(0) = 0$
- $F'(0) = I$, where I is the identity matrix.

Then $F(z) = z$

The answer to the question above in the hypersurface case is given by the theorem of Chern and Moser 1974

Theorem: Let M be a real hypersurface $M \subset \mathbb{C}^N$, $N = n+1$ then its automorphisms (biholomorphisms) are uniquely determined by their jets of order two at p provided that M is smooth, Levi non degenerate.

Definition of a jet of order 2 at p: all the derivatives up to 2 order

$$H(0) \quad \frac{\partial H}{\partial z}(0) \quad \frac{\partial H}{\partial w}(0) \quad \frac{\partial^2 H}{\partial z}(0) \quad \frac{\partial^2 H}{\partial z \partial w}(0)$$

$$\frac{\partial^2 H}{\partial w^2}(0)$$

Sketch of the proof in the case of $\mathbb{C}^2$.

$H: M \longrightarrow M$ $\quad H(0) = 0$. After a linear holomorphic change of coordinates we may assume that

$$H_1(z,w) = z + f(z,w)$$
$$H_2(z,w) = w + g(z,w)$$

Exercise: $g(z,w)$ does not contain terms in z^2

$$H_1(z,w) = \underset{\frac{1}{2}}{z} + \underbrace{f(z,w)}_{\text{weight} > \frac{1}{2}}$$

$$H_2(z,w) = \underset{1}{w} + \underbrace{g(z,w)}_{>1}$$

Remember $z \sim \frac{1}{2}$, $w \sim 1$ (4

$$f(z,w) = \sum_{\mu > \frac{1}{2}} f_\mu \qquad g(z,w) = \sum_{\mu > 1} g_\mu$$

Since $H : \Pi \to M$, we have the following equation

$$\cancel{Im}\ H_2(z,w) = \varphi\left(H_1(z,w), \overline{H_1(z,w)}, Re\, H_2(z,w)\right) \quad \Big|\ (z,w) \in M$$

$$= H_1 \overline{H_1} + \ldots$$

Writing $H_1 = z + f$, $H_2 = w + g$

$$\frac{w - \overline{w}}{2i} + \frac{g - \overline{g}}{2i} = (z+f)(\overline{z} + \overline{f}) + \ldots \Big|_M$$

$$\Rightarrow \frac{s + i\varphi(z,\overline{z},s)}{2i} - \left(\frac{s - i\varphi(z,\overline{z},s)}{2i}\right) + \frac{g - \overline{g}}{2i}$$

$$= \left(z + f(z, s + i\varphi(z,\overline{z},s))\right)\left(\overline{z} + \overline{f(z, s + i\varphi(z,\overline{z},s))}\right)$$

$$+ \ldots$$

$$\Rightarrow \varphi(z,\overline{z},s) + \frac{g - \overline{g}}{2i} = \left(z + f(z, s + i\varphi(z,\overline{z},s))\right)\left(\overline{z + f(z, s + i\varphi(z,\overline{z},s))}\right)$$

$$+$$

Using the Taylor expansion at $(z,\overline{z},s)$, we obtain

$$\varphi\left(z+f(z,s+i\varphi),\ \bar z+\overline{f(z,s+i\varphi)},\ s+\operatorname{Re} g(z,s+i\varphi(z,\bar z,s)\right)$$
$$=\varphi(z,\bar z,s)+\operatorname{Im} g(z,s+i\varphi(z,\bar z,s))$$

Identifying the terms of minimal degree $\mu_0 > 1$, one obtain

$$2\operatorname{Re} \bar z\, f^{(z+iz\bar z)}_{\mu_0-\frac{1}{2}}(z,s+iz\bar z)= \qquad (*)$$
$$\operatorname{Im} g_{\mu_0}(z,s+iz\bar z)$$

Exercice Show $(*)$

Now, let $Y = f_{\mu_0-\frac{1}{2}}\frac{\partial}{\partial z}+g_{\mu_0}\frac{\partial}{\partial w}$

we claim that $Y \in hol(M_0, 0)$, where M_0 is the hypersurface given by

$\operatorname{Im} w = z\bar z$! (forgetting about the higher order terms)

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Computing

$\mathrm{Re}\, \mathcal{Y}(\mathrm{Im}\, w - z\bar{z})$, we get

$$-f_{\mu_0-\frac{1}{2}}(z,w)\,\bar{z} + g_{\mu_0}(z,w)\frac{1}{2i}$$

$$-\mathrm{Re}\left(f_{\mu_0-\frac{1}{2}}\right)\bar{z} + \mathrm{Re}\left(g_{\mu_0}(\quad)\frac{1}{2i}\right)$$

$$= -\mathrm{Re}\left(f_{\mu_0-\frac{1}{2}}\right)\bar{z} + \frac{g_{\mu_0}\frac{1}{2i} - \frac{1}{2i}\overline{g_{\mu_0}}}{2}$$

$$= -\mathrm{Re}\, f_{\left(\mu_0-\frac{1}{2}\right)}\,\bar{z} + \frac{g_{\mu_0} - \bar{g}_{\mu_0}}{4i}$$

$$= -\mathrm{Re}\, f_{\mu_0-\frac{1}{2}}^{(z,\,s+iz\bar{z})}\,\bar{z} + \frac{\left(\mathrm{Im}\, g_{\mu_0}\right)^{(z,\,s+iz\bar{z})}}{2}$$

$$= \frac{-2\,\mathrm{Re}\, f_{0-\frac{1}{2}}\,\bar{z} + \mathrm{Im}\, g_{\mu_0}}{2}\Bigg|_{M_0}$$

Looking at *, we see that this last expression is zero, and therefore $\mathcal{Y} \in \mathrm{hol}(M_0, 0)$.

What are the consequences?

f_{μ_0} and g_{μ_0} have to be of weight order at most 2

If f_{μ_0} and $g_{\mu_0} > 2 \Longrightarrow$

$\| H(z,w) = \text{Identity} = (z,w)$

<u>Corollary:</u> If $H : M \longrightarrow M$

$K : M \longrightarrow M$

have the same jets up to 2, then $H(z,w) = K(z,w)$, which proves the theorem.

Proof. $H \circ K^{-1} = (z + \underbrace{\ldots}_{>2} \quad w + \underbrace{\ldots}_{>2})$

$\downarrow$ therefore 0 $\qquad \downarrow$ therefore 0

<u>Exercise:</u> Complete the details of the proof.

Final remarks

(8)

Here we have considered the Levi nondegenerate case, but there are theorem generalities to "Levi degenerate" hypersurface.

List of references.

Kolar Meylan Zaitsev for the hypersurface case

we may also consider manifolds of codimension >1 (hypersurface is codim 1)

Meylan, Beloshapka, etc