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A GENERALIZATION OF A MICROLOCAL VERSION OF BOCHNER'S THEOREM

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ABSTRACT. We prove a generalization of the microlocal version of Bochner's tube theorem obtained in Baouendi and Trèves [Indiana Univ. Math. J. 31 (1982), pp. 885–895]. The results provide a class of CR structures where CR functions extend holomorphically to a full neighborhood of a point which may be of infinite type.

1. Introduction

The classical extension theorem of Bochner states that if h is a holomorphic function on a tube $\mathbb{R}^m + \sqrt{-1}\Omega \subset \mathbb{C}^m$, where Ω is a domain in $\mathbb{R}^m$, then it extends holomorphically to the convex hull $\mathbb{R}^m + \sqrt{-1}\hat{\Omega}$. Local versions and different generalizations of Bochner's theorem have appeared in several works (see for example [1], [5], [10], [11], [12], [15], [16], [17], [21], [22], [24], [25] and the references therein). In [1] Baouendi and Trèves proved a microlocal version of Bochner's theorem. In particular, they obtained a necessary and sufficient condition for the analyticity of the solutions of the systems of vector fields under consideration. Their condition is also necessary for the smoothness of the solutions. However, in a subsequent paper [30], Ye constructed a counter example in $\mathbb{C}^2$ that showed that the condition in [1] is not sufficient in the smooth case. In this article we establish a sufficient condition for the holomorphic extendability of solutions which is more general than the one in [1]. As an application, we present a class of embedded CR structures where extendability to holomorphic functions in a full neighborhood or well defined wedges holds. These extendabilty results do not follow from the well known works [2] and [3]. The results proved here have applications to the C^∞ and analytic regularity of CR mappings between CR manifolds studied in the works [19], [20], [8], [9], [26], [27] and many articles cited in these latter papers. In the last section, we present another proof of the sufficiency condition for analytic hypoellipticity established in [1].

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2. Preliminaries and the main results

Let m and n be positive integers. We will denote by $x = (x_1, \ldots, x_m)$ and $t = (t_1, \ldots, t_n)$ variable points in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let V be a domain in

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$\mathbb{R}^n$ and
$$\phi(t) = (\phi_1(t), \dots, \phi_m(t))$$
a Lipschitz continuous mapping, $\phi : V \to \mathbb{R}^m$.

Let
$$Z_i(x,t) = x_i + \sqrt{-1}\phi_i(t),\ 1 \le i \le m$$
and consider the associated n complex vector fields on $\mathbb{R}^m \times V$ given by
$$L_j = \frac{\partial}{\partial t_j} - \sqrt{-1}\sum_{k=1}^{m} \frac{\partial \phi_k}{\partial t_j}(t)\frac{\partial}{\partial x_k},\ 1 \le j \le n.$$
The vector fields L_j define a system of linearly independent complex vector fields at each point in $\mathbb{R}^m \times V$. Clearly,
$$L_j Z_i = 0,\ 1 \le i \le m, 1 \le j \le n.$$
Let $U \subset \mathbb{R}^m$ be a domain and $\Omega = U \times V$. For $h = h(x,t)$ a Lipschitz continuous solution in Ω of the system of equations

$$L_j h = 0,\ 1 \le j \le n \tag{2.1}$$

and $t_0 \in V$, we will be interested in the analytic wave front set of the function
$$x \mapsto h(x,t_0) = h_{t_0}(x).$$
The reader is referred to the books [29] and [4] for the definition of microlocal analyticity and its characterization in terms of boundary values of holomorphic functions. We may assume $0 \in U$ and $t_0 = 0$. We may also assume that $\phi(0) = 0$ and $V = B_r(0)$ is a ball. Observe that when the mapping $\phi : V \to \mathbb{R}^m$ is an immersion, the L_j define a system of CR vector fields. We will give a sufficient condition for the microlocal analyticity of $h_0(x) = h(x,0)$.

The main result of this article is as follows:

Theorem 2.1. *Let $\xi^0 \in \mathbb{R}^m \setminus 0$. Assume that there is a sequence $t_j^* \in V \setminus 0$ converging to 0 such that for some fixed positive integer N,*

$$|\phi(t_j^*)|^{2N} < -\phi(t_j^*) \cdot \xi^0. \tag{2.2}$$

Then if h is any Lipschitz continuous solution of (2.1) in Ω, $(0,\xi^0)$ is not in the analytic wave front set of $h_0(x) = h(x,0)$.

Remark 2.1. When $N = 1$, this theorem follows from the main result in [1] which we recall here:

Theorem 2.2 (Theorem 1.1 in [1])**.** *Let $\xi^0 \in \mathbb{R}^m \setminus 0$ and assume there are $t^* \in U \setminus 0$ and a Lipschitz curve γ in V with 0 and t^* as its endpoints satisfying:*

$$-\phi(t^*) \cdot \xi^0 > 0, \tag{2.3}$$

$$\sup_{t \in \gamma} |\phi(t)| < r, \tag{2.4}$$

$$|\phi(t^*)|^2 \sup_{t \in \gamma} \phi(t) \cdot \xi^0 < [r^2 - \sup_{t \in \gamma} |\phi(t)|^2][-\phi(t^*) \cdot \xi^0]. \tag{2.5}$$

Then if h is any Lipschitz continuous solution of (2.1) in Ω, $(0,\xi^0)$ is not in the analytic wave front set of $h_0(x) = h(x,0)$.

The following two results on microlocal analyticity do not follow from the preceding two theorems.

Theorem 2.3. *Let $\xi^0 \in \mathbb{R}^m \setminus 0$ and assume there are $t^* \in U \setminus 0$ and a Lipschitz curve γ in V with 0 and t^* as its endpoints satisfying:*

$$-\phi(t^*)\cdot\xi^0 > 0, \tag{2.6}$$

$$2\sup_{t\in\gamma}|\phi(t)| < r, \tag{2.7}$$

$$8|\phi(t^*)|^4 \sup_{t\in\gamma}\phi(t)\cdot\xi^0 < [r^2 - 4\sup_{t\in\gamma}|\phi(t)|^2]^2[-\phi(t^*)\cdot\xi^0]. \tag{2.8}$$

Then if h is any Lipschitz continuous solution of (2.1) in Ω, $(0,\xi^0)$ is not in the analytic wave front set of $h_0(x) = h(x,0)$.

Theorem 2.4. *Let $\xi^0 \in \mathbb{R}^m \setminus 0$ and assume there are $t^* \in U \setminus 0$ and a Lipschitz curve γ in V with 0 and t^* as its endpoints satisfying:*

$$-\phi(t^*)\cdot\xi^0 > 0, \tag{2.9}$$

$$6\sup_{t\in\gamma}|\phi(t)| < r, \tag{2.10}$$

$$2\alpha|\phi(t^*)|^6 \sup_{t\in\gamma}\phi(t)\cdot\xi^0 < [r^2 - \sup_{t\in\gamma}|\phi(t)|^2]^3[-\phi(t^*)\cdot\xi^0], \tag{2.11}$$

$$\textit{where}\quad \alpha = (4+\sqrt{5})\left((4+\sqrt{5})^2 - 12(5+\sqrt{5})\right). \tag{2.12}$$

Then if h is any Lipschitz continuous solution of (2.1) in Ω, $(0,\xi^0)$ is not in the analytic wave front set of $h_0(x) = h(x,0)$.

Theorem 2.1 leads to the following corollary (see Corollary 1.2 in [1]) for solutions defined for all $x \in \mathbb{R}^m$.

Corollary 2.5. *Let $\xi^0 \in \mathbb{R}^m \setminus 0$. Assume there is $t^* \in V$ such that $-\phi(t^*)\cdot\xi^0 > 0$. If h is Lipschitz continuous solution of (2.1) in $\mathbb{R}^m \times V$, then $(0,\xi^0)$ is not in the analytic wave front set of h_0.*

The main tool used to prove the main theorem (Theorem 1.1 in [1]) in [1] is the FBI transform

$$\mathcal{F}u(x,\xi) = \int_{\mathbb{R}^m} e^{\sqrt{-1}(x-y)\cdot\xi - K|\xi||x-y|^2} u(y)dy,$$

where $u \in C_c^0(\mathbb{R}^m)$ is a continuous function of compact support and $K > 0$ is a suitably chosen constant. The assumption (2.5) which is quadratic in nature allowed a choice of K that shows the decay of the FBI transform which has a quadratic phase function. Indeed, we have:

$$\Re\left\{\sqrt{-1}(x-y-\sqrt{-1}\phi(t))\cdot\xi - K|\xi|(x-y-\sqrt{-1}\phi(t))^2\right\} = \phi(t)\cdot\xi - K|\xi|[|x-y|^2 - |\phi(t)|^2].$$

Thus given (2.5), one can choose $K > 0$ so that at t^*, for some $c > 0$

$$\phi(t^*)\cdot\xi - K|\xi|[|x-y|^2 - |\phi(t^*)|^2] < -c|\xi|, \tag{2.13}$$

for ξ in a conic neighborhood of ξ^0 and x near 0.

In our case, when $N > 1$, condition (2.2) does not lead to estimate (2.13) with the standard FBI. Instead, we use more general FBI transforms that were introduced in [6] which we next recall.

Consider a function $\Psi \in S(\mathbb{R}^m)$, $\int \Psi(x)dx = 1$, where $S(\mathbb{R}^m)$ denotes the Schwartz space of rapidly decaying functions. In [6] for any number $\lambda > 0$ we introduced a generalized FBI transform (with generating function Ψ and parameter λ) of a compactly supported continuous function $u \in C_c^0(\mathbb{R}^m)$ by the formula

$$\mathcal{F}_{\Psi,\lambda}u(x,\xi) = \int_{\mathbb{R}^m} e^{\sqrt{-1}(x-x')}\Psi(|\xi|^\lambda(x-x'))u(x')dx',$$

where $x, \xi \in \mathbb{R}^m$.

We also considered the generalized FBI transform of a compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^m)$ by viewing the integral as the duality bracket between a smooth function and a distribution with compact support. It was shown that microlocal smoothness (the C^∞ wave front set) can be characterized in terms of the rapid decay of the transforms $\mathcal{F}_{\Psi,\lambda}u$.

For the proof of Theorem 2.1, we will use these transforms with

$$\Psi(x) = ce^{-K|x|^{2k}},\ \lambda = \frac{1}{2k},$$

k a positive integer and c is chosen so that $\int_{\mathbb{R}^m} \Psi(x)dx = 1$. The real number $K > 0$ will be suitably chosen. That is, we will use the transforms

$$\mathcal{F}_k u(x,\xi) = \int_{\mathbb{R}^m} e^{\sqrt{-1}(x-x')\cdot\xi - K|\xi||x-x'|^{2k}}u(x')dx',$$

where $x, \xi \in \mathbb{R}^m$. We recall from [6] that each transform $\mathcal{F}_k u(x,\xi)$ characterizes the microlocal analyticity of a distribution u in terms of an exponential decay.

3. Proof of Theorem 2.1

We may assume $|\xi^0| = 1$. The number $K > 0$ will be chosen later. Let $g \in C_0^\infty(B_r(0))$, $g(x) \equiv 1$ for $|x| \le \frac{r}{2}$. For h a Lipschitz continuous solution of (2.1) in $\Omega = U \times V$ and $(x,\xi) \in \mathbb{R}^{2m}$, we will consider the integral

(3.1)
$$I^j(x,\xi) = \int_{\mathbb{R}^m}\int_{\gamma_j} e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t))\cdot\xi - K|\xi|[x-y-\sqrt{-1}\phi(t)]^{2k}} L(g(y)h(y,t))dtdy,$$

where $\gamma_j \subset V$ is a smooth curve joining 0 and t_j^*, and for $z \in \mathbb{C}^m$ we have used the notation

$$[z]^{2k} = \left(\sum_{j=1}^m z_j^2\right)^k,$$

$$Lf(y,t)dt = \sum_{j=1}^m L_j f(y,t)dt_j$$

is a one-form on V depending on y. An application of Stokes theorem leads to

$$I^j(x,\xi) = I_*^j(x,\xi) - I_0(x,\xi), \tag{3.2}$$

where

$$I_*^j(x,\xi) = \int_{\mathbb{R}^m} e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t_j^*))\cdot\xi - K|\xi|[x-y-\sqrt{-1}\phi(t_j^*)]^{2k}} g(y)h(y,t_j^*)dy,$$

$$I_0(x,\xi) = \int_{\mathbb{R}^m} e^{\sqrt{-1}(x-y)\cdot\xi - K|\xi||x-y|^{2k}} g(y)h_0(y)dy.$$

By Theorem A in [6], h_0 is microlocally analytic at $(0, \xi^0)$ if for some $c_1, c_2 > 0$,

$$|I_0(x,\xi)| \le c_1 e^{-c_2|\xi|}$$

for (x,ξ) in a conic neighborhood of $(0,\xi^0)$. In view of (3.2), it will suffice to prove similar estimates for $I^j(x,\xi)$ and $I^j_*(x,\xi)$ for some j.

Observe that for any t,

$$\left| e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t))\cdot\xi - K|\xi|[x-y-\sqrt{-1}\phi(t)]^{2k}} \right| = e^{-E(x,y,t,\xi)}$$

with

$$E(x,y,t,\xi) = -\phi(t)\cdot\xi + K|\xi|\Re[|x-y|^2 - |\phi(t)|^2 - 2\sqrt{-1}\langle x-y,\phi(t)\rangle]^k$$

$$= -\phi(t)\cdot\xi + K|\xi| \sum_{0\le 2s\le k} \binom{k}{2s} (|x-y|^2 - |\phi(t)|^2)^{k-2s}(-4)^s\langle x-y,\phi(t)\rangle^{2s}. \tag{3.3}$$

For each t fixed, we will determine a useful lower bound for the polynomial in x of degree $2k$ given by

$$P_{2k}(x-y,t) = \sum_{0\le 2s\le k} \binom{k}{2s} (|x-y|^2 - |\phi(t)|^2)^{k-2s}(-4)^s\langle x-y,\phi(t)\rangle^{2s} \tag{3.4}$$

and write

$$P_{2k}(x,t) = |x|^{2k} + \sum_{|\alpha|\le 2k} c_\alpha x^\alpha, \text{ with } c_\alpha = c_\alpha(t).$$

We will use the concepts and results of [13]. Given a non constant polynomial

$$f \in \mathbb{R}[x] = \mathbb{R}[x_1,\dots,x_m]$$

of degree $2d$, we decompose f into its homogeneous parts

$$f = f_0 + \dots + f_{2d},$$

where f_i is a form of degree i, $i = 0,\dots,2d$. Let

$$P_{2d,m} = \text{ the cone of all positive semidefinite forms of degree } 2d \text{ on } \mathbb{R}^m,$$

and

$$P^0_{2d,m} = \text{ the interior of } P_{2d,m}.$$

Write

$$f(x) = \sum_{|\alpha|\le 2d} f_\alpha\, x^\alpha$$

and let

$$S(f) = \{\alpha \in \mathbb{N}^m : f_\alpha \ne 0\} \setminus \{0, 2d\epsilon_1,\dots,2d\epsilon_m\},$$

where $\epsilon_i = (\delta_{i1},\dots,\delta_{im})$, and

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \ne j. \end{cases}$$

We denote $f_{2d\epsilon_i}$ by $f_{2d,i}$ for short. Thus f has the form

$$f(x) = f_0 + \sum_{\alpha\in S(f)} f_\alpha x^\alpha + \sum_{i=1}^m f_{2d,i}\, x_i^{2d}.$$

Let

$$\Delta(f) = \{\alpha \in S(f) : f_\alpha \, x^\alpha \text{ is not a square in } \mathbb{R}[x]\}$$
$$= \{\alpha \in S(f) : \text{ either } f_\alpha < 0 \text{ or } \alpha_i \text{ is odd for some } i \in \{1, \dots, m\}\}.$$

Recall next that a polynomial of one variable

$$q(s) = s^l - \sum_{i=0}^{l-1} a_i s^i,$$

where each $a_i \geq 0$ and at least one $a_i \neq 0$ has a unique positive root. This follows from Descartes' Rule of signs. We denote the positive root of such $q(s)$ by $C(q)$.

We now recall a theorem of [13] (Theorem 4.3 in [13]) which says: If $f_{2d} \in P^0_{2d,m}$ and $\epsilon > 0$ such that

$$f_{2d} - \epsilon \sum_{i=1}^{m} x_i^{2d} \in P_{2d,m},$$

then $f_0 - \lambda^{2d}$ is a lower bound of f, where

$$\lambda = C\left(s^{2d} - \sum_{i-0}^{2d-1} b_i s^i\right)$$

and

$$b_i = \frac{1}{2d}(2d-i)^{\frac{2d-i}{2d}} \epsilon^{\frac{-i}{2d}} \sum_{\alpha \in \Delta(f), |\alpha| = i} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}, \ \ 1 \leq i \leq 2d-1.$$

Recall that we are interested in a lower bound for

$$P_{2k}(x,t) = |x|^{2k} + \sum_{|\alpha|<2k} c_\alpha x^\alpha, \ c_\alpha = c_\alpha(t).$$

Since

$$|x|^{2k} \in P^0_{2k,m} \text{ and } |x|^{2k} - \sum_{i=1}^{m} x_i^{2k} \in P_{2k,m},$$

the preceding result tells us that we can take $\epsilon = 1$, and therefore,

$$P_{2k}(x,t) \geq (-1)^k |\phi(t)|^{2k} - \lambda^{2k}, \tag{3.5}$$

where

$$\lambda = C(s^{2k} - \sum_{i=0}^{2k-1} b_i s^i),$$

and

$$b_i = \frac{1}{2k}(2k-i)^{\frac{2k-i}{2k}} \sum_{\alpha \in \Delta, |\alpha| = i} |c_\alpha(t)| (\alpha^\alpha)^{\frac{1}{2k}}.$$

Note that we may assume there is $\alpha \in \Delta$ such that $c_\alpha(t) \neq 0$. Otherwise, we get an easy lower bound. Theorem 4.3 in [13] gives two other lower bounds, but we are quoting the one that will be useful for us.

We next estimate λ. It is mentioned without proof in [13] that

$$\lambda \leq 2 \max \{b_{2k-1}, b_{2k-2}^{\frac{1}{2}}, b_{2k-3}^{\frac{1}{3}}, \dots, b_0^{\frac{1}{2k}}\}.$$

(See Remark 3.1 below the end of the proof). To see this, let $j \in \{1, \dots, 2k\}$ satisfying

$$b_{2k-j}^{\frac{1}{j}} = \max \{b_{2k-1}, b_{2k-2}^{\frac{1}{2}}, b_{2k-3}^{\frac{1}{3}}, \dots, b_0^{\frac{1}{2k}}\}.$$

Then since

$$\lambda^{2k} = b_0 + b_1\lambda + b_2\lambda^2 + \ldots + b_{2k-1}\lambda^{2k-1},$$

$$\lambda^{2k} \leq b_{2k-j}^{\frac{2k}{j}} + b_{2k-j}^{\frac{2k-1}{j}}\lambda + b_{2k-j}^{\frac{2k-2}{j}}\lambda^2 + \ldots + b_{2k-j}^{\frac{1}{j}}\lambda^{2k-1}.$$

For some $c > 0$, let $\lambda = c\, b_{2k-j}^{\frac{1}{j}}$. Then

$$\begin{aligned} c^{2k} b_{2k-j}^{\frac{2k}{j}} &\leq b_{2k-j}^{\frac{2k}{j}}(1 + c + c^2 + \ldots + c^{2k-1}) \\ &= b_{2k-j}^{\frac{2k}{j}}\left(\frac{c^{2k}-1}{c-1}\right) \end{aligned}$$

and so

$$c^{2k} \leq \frac{c^{2k}-1}{c-1}.$$

If $0 < c \leq 2$, then $\lambda \leq 2b_{2k-j}^{\frac{1}{j}}$ and hence

$$\lambda \leq 2\max\{b_{2k-1}, b_{2k-2}^{\frac{1}{2}}, b_{2k-3}^{\frac{1}{3}}, \ldots, b_0^{\frac{1}{2k}}\}.$$

If $c > 2$, then the inequality $c^{2k} \leq \frac{c^{2k}-1}{c-1}$ leads to the contradiction

$$c^{2k}(c-2) \leq -1.$$

Thus

$$\lambda \leq 2\max\{b_{2k-1}, b_{2k-2}^{\frac{1}{2}}, b_{2k-3}^{\frac{1}{3}}, \ldots, b_0^{\frac{1}{2k}}\}. \tag{3.6}$$

Note also that since $b_i \geq 0$ for all i,

$$\lambda^{2k} \geq b_j\lambda^j \text{ for } 0 \leq j \leq 2k-1,$$

and so

$$\lambda \geq b_j^{\frac{1}{2k-j}}.$$

Therefore, we have

$$\begin{aligned} &\max\{b_{2k-1}, b_{2k-2}^{\frac{1}{2}}, b_{2k-3}^{\frac{1}{3}}, \ldots, b_0^{\frac{1}{2k}}\} \leq \lambda \\ &\qquad \leq 2\max\{b_{2k-1}, b_{2k-2}^{\frac{1}{2}}, b_{2k-3}^{\frac{1}{3}}, \ldots, b_0^{\frac{1}{2k}}\}. \end{aligned} \tag{3.7}$$

To estimate the b_i, we need to estimate each $c_\alpha = c_\alpha(t)$ for $|\alpha| = i$. Recall that for each t,

$$\begin{aligned} P_{2k}(x,t) &= \sum_{0\leq 2s\leq k} \binom{k}{2s}(|x|^2 - |\phi(t)|^2)^{k-2s}(-4)^s\langle x, \phi(t)\rangle^{2s} \\ &= |x|^{2k} + \sum_{|\alpha|\leq 2k} c_\alpha x^\alpha. \end{aligned}$$

It follows that there is a constant C_k depending only on k such that

$$|c_\alpha(t)| \leq C_k|\phi(t)|^{2k-|\alpha|}. \tag{3.8}$$

Hence there is a constant depending only on k, which we still denote by C_k such that

$$b_i \leq C_k|\phi(t)|^{2k-i} \text{ for } 0 \leq i \leq 2k-1. \tag{3.9}$$

From (3.6), (3.8), and (3.9), there is a constant $C = C(k) > 0$ such that

$$\lambda \leq 2\max\{b_j^{\frac{1}{2k-j}} : 0 \leq j \leq 2k-1\} \leq C|\phi(t)|.$$

By Theorem 4.3 in [13], we conclude that

$$P_{2k}(x,t) \geq -C|\phi(t)|^{2k},\ C = C(2k). \tag{3.10}$$

The bound (3.10) implies, with $k = N$,

$$\begin{aligned} E(x,y,t_j^*,\xi^0) &\geq -\phi(t_j^*)\cdot\xi^0 - CK|\phi(t_j^*)|^{2N}\ \forall j \\ &\geq |\phi(t_j^*)|^{2N}(1-CK) \text{ using (2.2).} \end{aligned}$$

Choose

$$0 < K < \frac{1}{2C}. \tag{3.11}$$

Then

$$-E(x,y,t_j^*,\xi^0) \leq -\frac{1}{2}|\phi(t_j^*)|^{2N}.$$

Since $E(x,y,t,\xi)$ is homogeneous of degree 1 in ξ, for each j, there is a conic neighborhood Γ_j of ξ^0 such that for some $c_1, c_2 > 0$:

$$|I_*^j(x,\xi)| \leq c_1 e^{-c_2|\xi|} \text{ for } \xi \in \Gamma_j. \tag{3.12}$$

Consider next the integral $I^j(x,\xi)$:

In the integrand of $I^j(x,\xi)$, since h is a solution and $g(y) \equiv 1$ for $|y| \leq \frac{r}{2}$, the integrand is supported in the set

$$\left\{y : |y| \geq \frac{r}{2}\right\}.$$

Since $\phi(0) = 0,\ t_j^* \to 0$, when $|y| \geq \frac{r}{2}$ and $|x| \leq \frac{r}{4}$, we get $c > 0$ and an integer j_0 such that

$$-E(x,y,t_{j_0}^*,\xi) \leq -c|\xi|\ \ \forall \xi \in \mathbb{R}^m$$

and hence for some $c_3, c_4 > 0$,

$$|I^{j_0}(x,\xi)| \leq c_3 e^{-c_4|\xi|} \text{ for } |x| \leq \frac{r}{4},\ \xi \in \mathbb{R}^m. \tag{3.13}$$

From (3.2),(3.12) and (3.13), we conclude that $(0,\xi^0)$ is not in the analytic wave front set of $h_0(x) = h(x,0)$.

Remark 3.1. The bound on the positive zero used in the proof is attributed by [13] to Exercise 4.6.2 in [23]. However, we have been unable to see the relation between the bound and the exercise.

Remark 3.2. Theorem 2.1 also holds for $\phi(t)$ assumed C^1 and solutions $h(x,t)$ which are measures. In that case, although the vector fields will have only continuous coefficients, one makes sense of a solution such as a measure and its trace at $t = 0$ using the results in [7].

Proof of Corollary 2.5. We first choose K as in (3.11) so that

$$-E(x,y,t^*,\xi^0) \leq -\frac{1}{2}|\phi(t^*)|^{2N}$$

which implies an exponential decay for

$$I_*(x,\xi) = \int_{\mathbb{R}^m} e^{-\sqrt{-1}(x-y-\sqrt{-1}\phi(t^*))\cdot\xi - K|\xi|[x-y-\sqrt{-1}\phi(t^*)]^{2k}} g(y)h(y,t^*)dy$$

for ξ in a conic neighborhood of ξ^0.

Let γ be any smooth curve joining 0 and t^*. We can then choose r large enough so that

$$I(x,\xi)=\int_{\mathbb{R}^m}\int_\gamma e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t))\cdot\xi-K|\xi|[x-y-\sqrt{-1}\phi(t)]^{2k}}L(g(y)h(y,t))dtdy$$

decays exponentially for $|x|\le\frac{r}{4}$, As before, the corollary follows from (3.2). $\square$

4. Proofs of Theorem 2.3 and Theorem 2.4

Consider next the lower bound

$$P_{2k}(x,t)\ge -c|\phi(t)|^{2k},\ c=c(k),$$

we had in (3.10) for our estimate of

$$E(x,y,t,\xi)=-\Re Q(x,y,t,\xi).$$

For the proofs below, we will take a closer look at $P_4(x,t)$ and $P_6(x,t)$.

Proof of Theorem 2.3. The polynomial $P_4(x,t)$ arises from the generalized FBI $\mathcal{F}_4u(x,\xi)$ and the corresponding

$$\begin{aligned}E(x,y,t,\xi)&=-\Re Q(x,y,t,\xi)\\&=-\xi\cdot\phi(t)+K|\xi|\Re[(x-y-\sqrt{-1}\phi(t))^4]\\&=-\xi\cdot\phi(t)+K|\xi|\Big[|x-y|^4-2|\phi(t)|^2|x-y|^2-4\langle x-y,\phi(t)\rangle^2+|\phi(t)|^4\Big],\end{aligned}$$

and so

$$P_4(x,t)=|x|^4-2|\phi(t)|^2|x|^2-4\langle x,\phi(t)\rangle^2+|\phi(t)|^4.$$

It is easy to see that for any t, the minimum of $P_4(x,t)$ is attained at $x=\sqrt{3}\phi(t)$ and the minimum value is $-8|\phi(t)|^4$.

It follows that

$$E(x,y,t^*,\xi^0)\ge-\xi^0\cdot\phi(t^*)-8K|\phi(t^*)|^4.$$

If we choose

$$0<K<\frac{-\xi^0\cdot\phi(t^*)}{8|\phi(t^*)|^4},$$

then the integral

$$I_*(x,\xi)=\int_{\mathbb{R}^m}e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t^*))\cdot\xi-K|\xi|[x-y-\sqrt{-1}\phi(t^*)]^4}g(y)h(y,t^*)dy$$

will satisfy the estimate

$$|I_*(x,\xi)|\le c_1e^{-c_2|\xi|}\quad\text{for some } c_1,c_2>0$$

and ξ in a conic neighborhood of ξ^0. To estimate the integral

$$I(x,\xi)=\int_{\mathbb{R}^m}\int_{\gamma_j}e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t))\cdot\xi-K|\xi|[x-y-\sqrt{-1}\phi(t)]^4}L(g(y)h(y,t))dtdy,$$

we choose $g(y)$ as before with $g(y) \equiv 1$ for $|y| \leq (1-\epsilon)r$ (ϵ to be determined) and consider for $|y| \geq (1-\epsilon)r$,

$$\begin{aligned}
&E(0,y,t^*,\xi^0)\\
&\quad= -\xi\cdot\phi(t) + K\left[|y|^4 - 2|\phi(t)|^2|y|^2 - 4\langle y,\phi(t)\rangle^2 + |\phi(t)|^4\right]\\
&\quad\geq -\xi\cdot\phi(t) + K\left[|y|^4 - 6|\phi(t)|^2|y|^2 + |\phi(t)|^4\right]\\
&\quad\geq -\xi\cdot\phi(t) + K\left[\left(|y|^2 - 3|\phi(t)|^2\right)^2 - 8|\phi(t)|^4\right]\\
&\quad\geq -\xi\cdot\phi(t) + K\left[\left(r^2 - 4|\phi(t)|^2\right)^2\right] \quad \text{using (2.7) and choosing } \epsilon \text{ small enough.}
\end{aligned}$$

We now choose K so that

$$\frac{\sup_\gamma \xi^0\cdot\phi(t)}{(r^2-4|\phi(t)|^2)^2} < K < \frac{-\xi^0\cdot\phi(t^*)}{8|\phi(t^*)|^4}.$$

Such a choice is possible because of assumption (2.8). Then for some $c_1, c_2 > 0$, x near the origin and ξ in a conic neighborhood of ξ^0,

$$|I(x,\xi)| \leq c_1 e^{-c_2|\xi|} \quad \text{for some } c_1, c_2 > 0$$

and hence $(0,\xi^0)$ is not in the analytic wave front set of the trace $h_0(x) = h(x,0)$. □

Proof of Theorem 2.4. $P_6(x,t)$ arises from the generalized FBI $\mathcal{F}_6 u(x,\xi)$ and the resulting

$$\begin{aligned}
&E(x,y,t,\xi)\\
&\quad= -\Re Q(x,y,t,\xi)\\
&\quad= -\xi\cdot\phi(t) + K|\xi|\Re[(x-y-\sqrt{-1}\phi(t))^6]\\
&\quad= -\xi\cdot\phi(t) + K|\xi|\left[(|x-y|^2-|\phi(t)|^2)^3 - 12(|x-y|^2-|\phi(t)|^2)((x-y)\cdot\phi(t))^2\right],
\end{aligned}$$

and thus

$$\begin{aligned}
P_6(x,t) &= |x|^6 - 3|\phi(t)|^2|x|^4 + 3|\phi(t)|^4|x|^2 - |\phi(t)|^6 - 12(|x|^2-|\phi(t)|^2)(x\cdot\phi(t))^2\\
&= |x|^6 - 3|\phi(t)|^2|x|^4 + (3|\phi(t)|^4 + 12(x\cdot\phi(t))^2)|x|^2\\
&\quad + 12(x\cdot\phi(t))^2|\phi(t)|^2 - |\phi(t)|^6.
\end{aligned}$$

For a fixed t, the minimum value of $P_6(x,t)$ is attained at a point x_0 where the gradient $\nabla_x P_6(x,t) = 0$.

Thus we have

$$6(|x_0|^2 - |\phi(t)|^2)^2 x_0 - 24(x\cdot\phi(t))^2 x_0 - 24(|x_0|^2 - |\phi(t)|^2)(x_0\cdot\phi(t))\phi(t) = 0,$$

which implies that

$$[(|x_0|^2-|\phi(t)|^2)^2 - 4(x_0\cdot\phi(t))^2]x = 4(|x_0|^2-|\phi(t)|^2)(x_0\cdot\phi(t))\phi(t). \tag{4.1}$$

If $(|x_0|^2 - |\phi(t)|^2)^2 - 4(x_0\cdot\phi(t))^2 = 0$, then (4.1) implies that

$$(|x_0|^2 - |\phi(t)|^2)(x_0\cdot\phi(t))\phi(t) = 0. \tag{4.2}$$

Since we are interested in $t^* = t$ where $\phi(t^*) \neq 0$, (4.2) in turn implies that

$$(|x_0|^2 - |\phi(t)|^2)(x_0 \cdot \phi(t)) = 0, \tag{4.3}$$

which by virtue of (4.1) implies that

$$|x_0| = |\phi(t)| \text{ and } x_0 \cdot \phi(t) = 0.$$

But then

$$P_6(x_0, t) = (|x_0|^2 - |\phi(t)|^2)^3 - 12(|x_0|^2 - |\phi(t)|^2)(x_0 \cdot \phi(t))^2 = 0.$$

But

$$P_6(0, t) = -|\phi(t)|^6 < P_6(x_0, t) = 0$$

since we are assuming that $\phi(t) \neq 0$. Thus

$$(|x_0|^2 - |\phi(t)|^2)^2 - 4(x_0 \cdot \phi(t))^2 \neq 0,$$

which by (4.1) means that the minimum occurs at a point x_0 of the form $c\,\phi(t)$ for some $c \in \mathbb{R}$. Therefore, the minimum of $P_6(x, t)$ is the same as that of the polynomial $f(c)$ of one variable given by

$$f(c) = |\phi(t)|^6[(c^2 - 1)^3 - 12(c^2 - 1)c^2].$$

Clearly, the minimum value of

$$h(c) = (c^2 - 1)^3 - 12(c^2 - 1)c^2$$

on $\mathbb{R}$ equals that of

$$g(s) = (s - 1)^3 - 12(s - 1)s \text{ on } [0, \infty).$$

It is easy to see that the minimum of g is attained at $s = 5 + \sqrt{5}$ and hence the minimum value of f and thus $P_6(x, t)$ is

$$\alpha|\phi(t)|^6 = (4 + \sqrt{5})\left((4 + \sqrt{5})^2 - 12(5 + \sqrt{5})\right)|\phi(t)|^6.$$

We therefore begin by choosing K to satisfy

$$0 < K < \frac{-\xi^0 \cdot \phi(t^*)}{\alpha|\phi(t^*)|^6},$$

and so the term

$$I_*(x, \xi) = \int_{\mathbb{R}^m} e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t^*))\cdot\xi - K|\xi|[x-y-\sqrt{-1}\phi(t^*)]^4} g(y)h(y, t^*)dy$$

will satisfy the estimate

$$|I_*(x, \xi)| \leq c_1 e^{-c_2|\xi|} \quad \text{for some } c_1, c_2 > 0$$

and ξ in a conic neighborhood of ξ^0.

To estimate the integral

$$I(x, \xi) = \int_{\mathbb{R}^m} \int_{\gamma_j} e^{\sqrt{-1}(x-y-\sqrt{-1}\phi(t))\cdot\xi - K|\xi|[x-y-\sqrt{-1}\phi(t)]^4} L(g(y)h(y, t))dtdy,$$

we choose $g(y)$ as before with $g(y) \equiv 1$ for $|y| \leq (1 - \epsilon)r$ (ϵ to be determined) and consider for $|y| \geq (1 - \epsilon)r$, the function $E(0, y, t^*, \xi^0)$. Observe that because of inequality (2.10),

$$|y \cdot \phi(t)| < \frac{1}{5}\left(|y|^2 - |\phi(t)|^2\right) \quad \text{for } \epsilon \text{ sufficiently small}$$

and hence

$$\begin{aligned} E(0,y,t^*,\xi^0) &= -\xi^0\cdot\phi(t) + K\left[(|y|^2-|\phi(t)|^2)^3 - 12(|y|^2-|\phi(t)|^2)(y\cdot\phi(t))^2\right] \\ &\geq -\xi^0\cdot\phi(t) + K\left[(|y|^2-|\phi(t)|^2)^3 - \frac{12}{25}(|y|^2-|\phi(t)|^2)^3\right] \\ &\geq -\xi^0\cdot\phi(t) + \frac{K}{2}\left((|y|^2-|\phi(t)|^2)\right)^3. \end{aligned}$$

We therefore choose K so that

$$2\frac{\sup_\gamma \xi^0\cdot\phi(t)}{\left(r^2-\sup_\gamma|\phi(t)|^2\right)^3} < K < \frac{-\xi^0\cdot\phi(t^*)}{\alpha|\phi(t^*)|^6}.$$

Such a choice is made possible by assumption (2.11). Then for some $c_1, c_2 > 0$, x near the origin and ξ in a conic neighborhood of ξ^0,

$$|I(x,\xi)| \leq c_1 e^{-c_2|\xi|} \quad \text{for some } c_1, c_2 > 0$$

and hence $(0,\xi^0)$ is not in the analytic wave front set of the trace $h_0(x) = h(x,0)$. □

5. Examples

Let S be the complement of the union

$$\bigcup_{n=0}^{\infty}\left(\frac{-1}{2^{4n+1}}, \frac{-1}{2^{4n+3}}\right) \cup \bigcup_{n=0}^{\infty}\left(\frac{1}{2^{4n+3}}, \frac{1}{2^{4n+1}}\right)$$

in the real line $\mathbb{R}$ and choose $f \in C^\infty(\mathbb{R})$ such that

$$f^{-1}(0) = S \text{ and } f \geq 0.$$

Such an f exists because S is a closed set.

Define the function ϕ_1 by

$$\phi_1(t) = \begin{cases} f(t), & \frac{1}{2^{4n+3}} \leq |t| \leq \frac{1}{2^{4n+1}} \\ -h(t)f(\frac{t}{2^2}), & \frac{1}{2^{4n+5}} \leq |t| \leq \frac{1}{2^{4n+3}} \end{cases}$$

where

$$h(t) = \begin{cases} e^{\frac{-1}{t}}, \; t > 0 \\ 0, \; t \leq 0. \end{cases}$$

Let

$$\phi(t) = (\phi_1(t), \phi_2(t)),$$

$\phi_2(t)$ to be determined later. If $\xi^0 = (1,0)$, then $\xi^0\cdot\phi(t) = \phi_1(t)$.

Suppose $\xi^0\cdot\phi(t^*) < 0$ for some t^*. Then there is n such that

$$\frac{1}{2^{4n+5}} < t^* < \frac{1}{2^{4n+3}}$$

and so

$$\phi_1(t^*) = -e^{\frac{-1}{t^*}}\left[f\left(\frac{t^*}{2^2}\right)\right]^2.$$

Let $\gamma = [0, t^*]$. Clearly,

$$\sup_\gamma \phi(t)\cdot\xi^0 = \sup_\gamma \phi_1(t) \geq \phi_1\left(\frac{t^*}{2^2}\right) = f\left(\frac{t^*}{2^2}\right)$$

since

$$\frac{1}{2^{4(n+1)+3}} < \frac{t^*}{2^2} < \frac{1}{2^{4(n+1)+1}}.$$

Therefore,

$$|\phi(t^*)|^2 \sup_{\gamma} \phi(t) \cdot \xi^0 \geq |\phi(t^*)|^2 f\left(\frac{t^*}{2^2}\right) \geq |\phi_2(t^*)|^2 f\left(\frac{t^*}{2^2}\right).$$

Hence if condition (1.9) in [1] (conditions (2.5) in this paper) holds, then

$$|\phi_2(t^*)|^2 f\left(\frac{t^*}{2^2}\right) < \left(r^2 - \sup_{\gamma} |\phi(t)|^2\right) e^{\frac{-1}{t^*}} \left[f\left(\frac{t^*}{2^2}\right)\right]^2,$$

that is,

$$|\phi_2(t^*)|^2 < r^2 e^{\frac{-1}{t^*}} f\left(\frac{t^*}{2^2}\right),$$

an inequality that may not hold when we work on

$$\Omega = \{(x,t) : |x| < r, |t| < T\}.$$

For example, such an inequality will not hold if we choose

$$\phi_2(t) = e^{\frac{-1}{3t}} \sqrt{f\left(\frac{t}{2^2}\right)}$$

and $f(t)$ is the square of a smooth function. Thus the Baouendi-Trèves condition is not met even for any sequence t_k^* that converges to 0.

However,

$$|\phi_2(t)|^4 = \left[f\left(\frac{t^*}{2^2}\right)\right]^2 e^{\frac{-4}{3t^*}} < -\phi_1(t^*) = e^{\frac{-1}{t^*}} \left[f\left(\frac{t^*}{2^2}\right)\right]^2$$

and so (2.2) holds for a sequence t_k^* that converges to 0.

We next give examples of Liptschitz and C^∞ hypersurfaces Σ in $\mathbb{C}^{n+1}$ where CR functions near the origin in Σ extend to holomorphic functions in a full neighborhood of the origin.

Let $z_j = x_j + iy_j$, $1 \leq j \leq n$ denote the coordinates in $\mathbb{C}^n$, and $\psi(z)$ a holomorphic function defined near the origin in $\mathbb{C}^n$, $\psi(0) = 0$ and $d\psi(0) = 0$ (or $d\psi(0)$ small enough). Let $f = f(y_1, \ldots, y_n)$ be a Liptschitz function near the origin in $\mathbb{R}^n$, $f(0) = 0$. Consider the hypersurface given by

$$\Sigma = \{(z_1, \ldots, z_n, s + i(\Re\psi(z) + f(y))\} \quad s \in \mathbb{R}.$$

Assume $|f(y)| \leq C|y|^2$ for some $C > 0$ and that there exist two sequences $\{p_\nu^*\}$ and $\{q_\nu^*\}$ in $\mathbb{R}^n$ both converging to 0 such that for some integer $k \geq 1$,

$$f(p_\nu^*) \geq |p_\nu^*|^{2k} \quad \text{and} \quad -f(q_\nu^*) \geq |q_\nu^*|^{2k} \quad \forall \nu.$$

Then any CR function defined near the origin on Σ extends to a holomorphic function in a full neighborhood of the origin in $\mathbb{C}^{n+1}$. In [18], Hounie and Tavares showed that after a biholomorphism and change of coordinates, the Lewy hyperquadrics

$$(z_1, \ldots, z_n, s + i\sum_{j=1}^{n} \epsilon_j |z_j|^2), \quad \epsilon_j \in \{-1, 1\}$$

become tube structures. We use the same idea to show first that Σ also is equivalent to a tube structure. Indeed, let

$$\Phi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$$

be the biholomorphic map defined by $w = \Phi(z)$ where $w_j = z_j$, $1 \leq j \leq n$, and $w_{n+1} = z_{n+1} - i\psi(z_1, \dots, z_n)$. Then

$$\Sigma' = \Phi(\Sigma) = (z_1, \dots, z_n, s + \Im\psi(z) + if(y)).$$

Define new coordinates

$$\begin{cases} x'_j = x_j, \ 1 \leq j \leq n, \\ x'_{n+1} = s + \Im\psi(z_1, \dots, z_n) \\ t'_j = y_j, \ 1 \leq j \leq n. \end{cases}$$

Dropping the primes, in the new coordinate system, we get a tube CR structure with first integrals

$$Z_j(x,t) = x_j + it_j, \ 1 \leq j \leq n, \quad Z_{n+1} = x_{n+1} + if(t) \quad \text{near } (0,0) \in \mathbb{R}^{n+1} \times \mathbb{R}^n.$$

We have

$$\Phi(t_1, \dots, t_n) = (t_1, \dots, t_n, f(t_1, \dots, t_n)).$$

It is clear that for any $\xi^0 \in \mathbb{R}^{n+1} \setminus 0$, there is an integer N and a sequence t_j^* in $\mathbb{R}^n$ converging to 0 such that $|\phi(t_j^*)|^{2N} < -\phi(t_j^*) \cdot \xi^0$. By Theorem 2.1, it follows that the trace $h_0(x)$ of any solution $h(x,t)$ is real analytic at the origin, and hence for some holomorphic function H defined near the origin in $\mathbb{C}^{n+1}$, $h(x,t) = H(Z_1(x,t), \dots, Z_{n+1}(x,t))$.

The function $f(y)$ can be chosen so that this extendability result does not follow from the results in [2]. Indeed, for example, one can take

$$f(y) = f(y_1) = y_1^{2k+1}\sin(g(y_1)), \ k \text{ a positive integer}$$

and

$$g(y_1) = \begin{cases} e^{\frac{-1}{y_1}}, \ y_1 > 0 \\ 0, \ y_1 \leq 0. \end{cases}$$

Observe that the resulting CR structure is then of infinite type at the origin. Likewise, one can give similar examples for higher codimensions where extendability to a holomorphic function holds and the results in [3] do not apply.

If we take $f(y) = f(y_1) = y_1^{2k+2}\sin\left(\frac{1}{y_1^{2k}}\right)$, $k \geq 1$, we get a C^1 CR structure with the vector fields having continuous coefficients where solutions which are apriori assumed to be measures extend holomorphically to a full neighborhood. Indeed, we can use the version of the approximation theorem of Baouendi and Trèves for vector fields with continuous coefficients proved in [7]. In that version, the authors showed how to define in a natural way the trace of a measure solution on a maximally real submanifold of an embedded C^1 CR manifold. Moreover, the version of the approximation theorem of Baouendi and Trèves proved in [7] implies that if the trace of a measure solution is real analytic on a real analytic maximally real submanifold, then the solution itself equals a holomorphic function of the first integrals. For microlocal regularity in elliptic directions when the vector fields have low regularity such as C^1, something that is also used here, we refer the reader to [14].

6. Remarks on the analytic case

In [1] Baouendi and Trèves established a necessary and sufficient condition for the analyticity of all solutions of the system of equations (2.1). In this section we will briefly present another proof for the sufficiency part (which is the difficult part) of their result.

Let m and n be positive integers. We will continue to denote by $x = (x_1, \dots, x_m)$ and $t = (t_1, \dots, t_n)$ variable points in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let V be a domain in $\mathbb{R}^n$ and

$$\varphi(t) = (\varphi_1(t), \dots, \varphi_m(t))$$

a real analytic mapping, $\varphi : V \to \mathbb{R}^m$.

Let

$$Z_i(x,t) = x_i + \sqrt{-1}\varphi_i(t),\ 1 \le i \le m$$

and consider the associated n complex vector fields on $\mathbb{R}^m \times V$ given by

$$L_j = \frac{\partial}{\partial t_j} - \sqrt{-1}\sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial t_j}(t)\frac{\partial}{\partial x_k},\ 1 \le j \le n.$$

Let $\Omega = \mathbb{R}^m \times V$. We will consider continuous solutions $h = h(x,t)$ of the system of equations

$$L_j h = 0,\ 1 \le j \le n \tag{6.1}$$

on open subsets of Ω.

We denote by $\mathcal{L}$ the system of vector fields $L_1, \dots, L_n$.

Definition 6.1. We say that $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if for any distribution u, whenever $L_j u$ $(j = 1, \dots, n)$ is real analytic in a neighborhood of (x_0, t_0), u itself is real analytic in a possibly smaller neighborhood of (x_0, t_0). We say that $\mathcal{L}$ is analytic hypoelliptic on a subset of $\mathbb{R}^m \times V$ if $\mathcal{L}$ is analytic hypoelliptic at each point of the subset.

Since the coefficients of the L_j are independent of x, it is clear that $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if and only if $\mathcal{L}$ is analytic hypoelliptic on $\mathbb{R}^m \times \{t_0\}$.

We will use the following result from [1]:

Proposition 6.2. *The system $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if and only if for every distribution h defined in some neighborhood of (x_0, t_0) which is a solution, the distribution $x \longmapsto h(x, t_0)$ is real analytic in some neighborhood of x_0.*

We recall the characterization of analytic hypoellipticity proved by Baouendi and Trèves (Theorem 2.1 in [1]):

Theorem 6.3. *The system $\mathcal{L}$ is analytic hypoelliptic at $(x_0, t_0) \in \mathbb{R}^m \times V$ if and only if for every $\xi \in \mathbb{R}^m \backslash \{0\}$, t_0 is not a local extremum of the function $t \longmapsto \varphi(t) \cdot \xi$.*

In the work [1], the sufficiency of the extremum condition for analytic hypoellipticity was proved using a consequence of Hironaka's theorem. Our objective here is to indicate that instead of using Hironaka's result, one can also employ the much easier curve selection lemma of Milnor which we next recall from [28]:

Let $A \subset \mathbb{R}^m$ be a real algebraic set, and let $B \subset \mathbb{R}^m$ be an open set defined by finitely many polynomial inequalities:

$$B = \{x \in \mathbb{R}^m : g_1(x) > 0, \dots, g_l(x) > 0\}.$$

Lemma 6.4 (Lemma 3.1 in [28])**.** *If $A \cap B$ contains points arbitrary close to the origin (that is $0 \in \overline{A \cap B}$) then there exists a real analytic curve*

$$p : [0, \epsilon) \to \mathbb{R}^m$$

with $p(0) = 0$ and with $p(t) \in A \cap B$ for $t > 0$.

With some minor modifications, the proof of Milnor's lemma also works for A a real analytic variety and the functions g_j real analytic on some neighborhood of the origin in $\mathbb{R}^m$. The curve $p(t) \in A \cap B$ for $t > 0$ can then be used to prove the sufficiency part of the analytic hypoellipticity result of Baouendi and Trèves. The details of the proof are as in those of Theorem 2.1 in [1].

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